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Holomorphic matrix models

Calin I. Lazaroiu

Insitut für Physik, Humboldt Universität zu Berlin
Invalidenstrasse 110, Berlin, Germany
E-mail: calin@physik.hu-berlin.de

Abstract: This is a study of holomorphic matrix models, the matrix models which underlie the conjecture of Dijkgraaf and Vafa. I first give a systematic description of the holomorphic one-matrix model. After discussing its convergence sectors, I show that certain puzzles related to its perturbative expansion admit a simple resolution in the holomorphic set-up. Constructing a 'complex' microcanonical ensemble, I check that the basic requirements of the conjecture (in particular, the special geometry relations involving chemical potentials) hold in the absence of the hermiticity constraint. I also show that planar solutions of the holomorphic model probe the entire moduli space of the associated algebraic curve. Finally, I give a brief discussion of holomorphic ADE models, focusing on the example of the A_2 quiver, for which I extract explicitly the relevant Riemann surface. In this case, use of the holomorphic model is crucial, since the hermitean approach and its attending regularization would lead to a singular algebraic curve, thus contradicting the requirements of the conjecture. In particular, I show how an appropriate regularization of the holomorphic A_2 model produces the desired smooth Riemann surface in the limit when the regulator is removed, and that this limit can be described as a statistical ensemble of 'reduced' holomorphic models.

Keywords: Supersymmetry and Duality, Matrix Models.

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1. Introduction

In the seminal paper [1], Dijkgraaf and Vafa proposed a beautiful conjecture relating matrix models to closed string theory on certain noncompact Calabi-Yau spaces. In its strongest form, this is meant as a relation between the partition function of a certain matrix model and the partition function of Kodaira-Spencer theory [2] on a dual noncompact Calabi-Yau threefold. The large-N limit of this relation also leads to a matrix model description of corrections to the Veneziano-Yankielowicz potential of certain $N = 1$ supersymmetric field theories [1, 3] (moreover, the VY potential itself can be recovered from the non-perturbative part of the large-N matrix integral). In fact, there are now independent derivations of the planar version of this conjecture by purely field-theoretic methods [4, 5, 6, 7], as well as first tests of it beyond the large-N limit [8, 9]. More recent work on the subject can be found in [10]–[17].

As already pointed out in [1], a proper formulation of the conjecture should be given in terms of 'holomorphic matrix models', namely some version of matrix models involving 'contour' integrals in a space of complex matrices. This is very natural if one remembers that the conjecture was originally derived by considering the worldvolume theory of certain topological B-type branes, which is described by a reduction of holomorphic Chern-Simons theory [18]. Since the latter is formulated without reference to a metric, the proper description of the resulting matrix model should not involve a hermiticity constraint. This expectation is borne out by the fact that the conjecture describes relations in the chiral ring of the dual field theory [6], and the latter is constrained by holomorphy.

In [1], the authors chose the pragmatic approach of formulating the conjecture in terms of hermitean matrix models, while pointing out that an appropriate holomorphic formulation should be given. The purpose of the present work is to initiate a systematic study of such holomorphic models, and show how they naturally tie up some loose ends encountered in various investigations and extensions of the conjecture.

If naively taken at face value, the hermitean formulation leads to various problems, some of which were already encountered (and overcome by a pragmatic prescription) in work aimed at testing it [8]. Among such issues, one can list the following:

(a) Hermitean one-matrix models with generic polynomial potentials of odd degree (and complex coefficients) are ill-defined, since the real part of such potentials is not bounded from below along the real axis. However, such models are naturally required by the conjecture, which is supposed to apply without constraints on the degree of the potential. This issue was encountered in the work of [8], while performing a one-loop test of the conjecture for a *cubic* potential. As shown in section 3 and appendix B, the pragmatic approach followed in [8] admits a natural justification in the holomorphic setup.

(b) The large-N spectral density $\omega_0(z) = \mathrm{tr}\,\frac{1}{z-M}$ of a hermitean one-matrix model (with matrix M) can only have cuts along the real axis. This means that the cuts of the hyperelliptic Riemann surface of [1] would be constrained to lie on the real line. In the hermitean formulation, the conjecture would then imply that the dual Calabi-Yau space is constrained in a similar manner. Moreover, the hermitean formulation leads to numerous problems in matching parameters and moduli spaces, since a polynomial of degree n always has n complex roots, but need not have n *real* roots. As clear from the work of [1] (and occasionally pointed out in subsequent work by the same authors), what is needed is a holomorphic extension of the hermitean matrix model which would allow for an enlarged family of planar limits — namely, such a model should produce solutions whose large-N eigenvalue distributions can be supported along arbitrary one-dimensional curve segments in the complex plane.

In the present note, I show how these and related issues are solved by a direct analysis of holomorphic matrix models, whose construction follows a suggestion already made in [1]. In sections 2 and 3, I explicitate the non-perturbative definition of such models[1] and study their convergence sectors, thereby refining one side of the conjecture. I also show (in appendix B) that issues like those encountered in [8] admit a natural resolution in this framework (which *does* recover the prescription used in that paper). Section 4 extracts the loop equations, equations of motion and the planar limit of such models, showing that most of the standard analysis employed for the hermitean model carries through with certain modifications. Consideration of the large-N limit leads to the algebraic curve of [1], which is now *unconstrained* by any conditions on the location of the cuts. In fact, one can give a 'reconstruction theorem', which ensures that the holomorphic model probes the entire moduli space of this algebraic curve. This shows explicitly how such models solve the second issue listed above. Up to some details, the reconstruction result boils down to the

[1]I should state from the outset that holomorphic matrix models are not the same as the so-called 'complex matrix models' [19] sometimes encountered in the matrix model literature. Rather, they are a sort of 'square roots' of such models.

well-known relation between the Riemann problem and singular integral equations. From this perspective, holomorphic matrix models give a sort of 'quantization' of the classical Riemann problem.

The validity of the conjecture rests crucially on the special geometry relations mentioned in [1], a more detailed justification of which was latter given in [20] (though only for the hermitean case). To give a clear proof of such relations at the holomorphic level, section 5 constructs a 'complex' microcanonical ensemble by introducing chemical potentials and performing a Legendre transform, thereby obtaining a generating functional which depends on *averaged* filling fractions; this allows one to show that the special geometry relations of [1] are a direct consequence of standard equations expressing chemical potentials in terms of the microcanonical generating function. In particular, these relations give the finite N analogue of the special geometry constraints.

Section 6 considers holomorphic ADE models, focusing on a detailed analysis of the A_2 model. In this case, the problems one encounters by working naively with a hermitean constraint are considerably more severe than in the one-matrix case. In fact, the hermitean approach of [21] (with its attending regularization) would lead to a singular curve, which would always be constrained to have a certain number of double points. This would completely miss some of the gauge theory physics extracted in [22, 23] More precisely, such a constraint would force certain filling fractions to be identically zero, thereby contradicting the existence of the gaugino condensates obtained in [22]. As shown in section 6, this issue is resolved quite naturally in the *holomorphic* A_2 model, by considering a regularization which is natural in that set-up. When removing the regulator, one obtains an ensemble of reduced holomorphic models, whose planar limit allows for non-vanishing values of *all* filling fractions. It is the large-N limit of this ensemble which is relevant for the conjecture of [1].

2. Construction of holomorphic one-matrix models and their eigenvalue representation

Following a suggestion made in [1], we start by constructing holomorphic one-matrix models as multidimensional 'contour' integrals in a space of complex matrices. The construction is quite similar to that of a hermitean model, except that the hermiticity constraint is replaced by a more general condition on the eigenvalues. After defining the model and studying its gauge-invariance, we extract an eigenvalue representation by choosing an appropriate multidimensional 'contour'. With this choice, one obtains an integral expression for the partition function which is formally identical to that of a hermitean model, except that integration is performed over eigenvalues lying along an open path in the complex plane.

2.1 The partition function

Fixing a positive integer N, we let $\mathrm{Mat}_N(\mathbb{C})$ denote the set of all $N \times N$ complex matrices. For any such matrix M, we let $p_M(\lambda) = \det(\lambda I - M)$ denote its characteristic polynomial. Define a subset $\mathcal{M}$ of $\mathrm{Mat}_N(\mathbb{C})$ as follows:

$$\mathcal{M} := \{M \in \mathrm{Mat}_N(\mathbb{C}) \mid p_M(\lambda) \text{ has distinct roots}\}\,. \tag{2.1}$$

This is an open submanifold of $\mathrm{Mat}_N(\mathbb{C})$, consisting of matrices which are diagonalizable by general linear transformations: for every M in $\mathcal{M}$, there exists an $S \in \mathrm{GL}(N, \mathbb{C})$ such that:

$$SMS^{-1} = D := \mathrm{diag}(\lambda_1 \dots \lambda_N)\,, \tag{2.2}$$

where λ_j are the roots of $p_M(\lambda)$. The later coincide with the eigenvalues of M, so that its spectrum is given by:

$$\sigma(M) = \{\lambda_1 \dots \lambda_N\}\,. \tag{2.3}$$

Consider a connected, noncompact and boundary-less submanifold Γ of $\mathcal{M}$ of real dimension equal to the complex dimension of $\mathcal{M}$:

$$\dim_{\mathbb{R}} \Gamma = \dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{C}} \mathrm{Mat}_N(\mathbb{C}) = N^2\,. \tag{2.4}$$

Also consider the standard holomorphic symplectic form on $\mathrm{Mat}_N(\mathbb{C})$:

$$w = \wedge_{i,j} dM_{ij}\,. \tag{2.5}$$

The sign of the right hand side of course depends on the ordering of pairs $(i, j) \in \{1, \dots, N\}$ $\times\{1, \dots, N\}$, and we shall use the lexicographic ordering:

$$w = dM_{11} \wedge \dots \wedge dM_{1N} \wedge dM_{21} \wedge \dots \wedge dM_{2N} \wedge \dots \wedge dM_{N1} \wedge \dots \wedge dM_{NN}\,. \tag{2.6}$$

This is implicit in all such expressions encountered below.

Fixing a polynomial $W(z) = \sum_{m=0}^{n} t_m z^m$ with complex coefficients, we define the *holomorphic one-matrix model* by the partition function:

$$\tilde{\mathcal{Z}}_N(\Gamma, t) = \frac{1}{\mathcal{N}} \int_\Gamma w e^{-N \,\mathrm{tr}\, W(M)}\,, \tag{2.7}$$

where $\mathcal{N}$ is a normalization factor to be fixed below.

2.2 Gauge invariance

Consider the $\mathrm{GL}(N, \mathbb{C})$ action τ on $\mathrm{Mat}_N(\mathbb{C})$ given by similarity transformations:

$$\tau(S)M := SMS^{-1}\,, \qquad S \in \mathrm{GL}(N, \mathbb{C})\,. \tag{2.8}$$

This clearly stabilizes $\mathcal{M}$; in particular, note that the characteristic polynomial of M is τ-invariant:

$$p_{\tau(S)M}(\lambda) = p_M(\lambda)\,. \tag{2.9}$$

The action $\mathcal{S}(M) = N \,\mathrm{tr}\, W(M)$ of our model is obviously invariant with respect to (2.8). The same is true of the holomorphic measure w: since $[\tau(S)M]_{ij} = a_{ij,kl} M_{kl}$, with $a_{ij,kl} = S_{ik}(S^{-1})^T_{jl}$ (i.e. $a = S \otimes (S^{-1})^T$), we have $\det(a) = 1$ and:

$$\tau(S)^*(w) = w\,. \tag{2.10}$$

It follows that the model admits the $\mathrm{GL}(N, \mathbb{C})$ gauge-invariance (2.8), provided that one chooses the integration manifold Γ such that it is stabilized by the gauge-group action.

Observation. Choosing S to be a permutation matrix (i.e. $S_{ij} = \delta_{j\sigma(i)}$ with σ a permutation on N elements), equation (2.10) shows that the holomorphic measure (2.5) is invariant under joint permutations of indices:

$$\begin{aligned} &dM_{11} \wedge \cdots \wedge dM_{1N} \wedge dM_{21} \wedge \cdots \wedge dM_{2N} \wedge \cdots \wedge dM_{N1} \wedge \cdots \wedge dM_{NN} = \\ &\quad = dM_{\sigma(1)\sigma(1)} \wedge \cdots \wedge dM_{\sigma(1)\sigma(N)} \wedge dM_{\sigma(2)\sigma(1)} \wedge \cdots \wedge dM_{\sigma(2)\sigma(N)} \wedge \cdots \\ &\qquad \cdots \wedge dM_{\sigma(N)\sigma(1)} \wedge \cdots \wedge dM_{\sigma(N)\sigma(N)} \,, \end{aligned} \tag{2.11}$$

a relation which can also be checked directly.

2.3 Eigenvalue representation

One can derive an eigenvalue representation of (2.7) as follows. Let $\gamma : \mathbb{R} \to \mathbb{C}$ be an open curve in the complex plane, which we assume to be immersed and without self-intersections. We shall take Γ to be the following subset of $\mathcal{M}$:

$$\Gamma(\gamma) := \{M \in \mathcal{M} \mid \sigma(M) \subset \gamma\} \,, \tag{2.12}$$

which obviously satisfies (2.4). We denote $\tilde{\mathcal{Z}}_N(\Gamma(\gamma), t)$ by $\tilde{\mathcal{Z}}_N(\gamma, t)$. Then the integral over gauge-group orbits can be performed as explained in appendix A, with the result:

$$\tilde{\mathcal{Z}}_N(\gamma, t) = \frac{1}{\mathcal{N}}(-1)^{N^2(N-1)/2} \frac{1}{N!} \,\mathrm{hvol}(H) \mathcal{Z}_N(\gamma, t) \,, \tag{2.13}$$

where $\mathrm{hvol}(H)$ is the 'holomorphic volume' of the complex homogeneous space $H = \mathrm{GL}(N, \mathbb{C})/(\mathbb{C}^*)^N$ (see appendix A) and

$$\mathcal{Z}_N(\gamma, t) = \int_\gamma d\lambda_1 \cdots \int_\gamma d\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^N W(\lambda_j)} \,. \tag{2.14}$$

is the *eigenvalue representation* of our model. The holomorphic volume $\mathrm{hvol}(H)$ will be discarded (together with the sign prefactors) by choosing $\mathcal{N} = (-1)^{N^2(N-1)/2} \frac{1}{N!} \mathrm{hvol}(H)$ in (2.7). Expression (2.14) is formally identical with the eigenvalue representation of the hermitean matrix model, except that the eigenvalue integral is performed along the contour γ in the complex plane. The pragmatic reader can take (2.14) as the definition of our model.

3. Convergence sectors

It is clear from expression (2.14) that convergence of our partition function depends on the choice of γ. In this section, we shall characterize the 'good' choices of γ in terms of certain asymptotic sectors of the model, described in terms of cones partitioning the complex plane. Such cones can be identified by performing an elementary analysis of the behavior of the integrand. As we shall see below, this allows us to make non-perturbative sense of models with polynomial potentials of odd degree.

To extract the relevant behavior, let $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $t_j = r_j e^{-i\theta_j}$ with $r_j > 0$ and $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$. Then the potential takes the form:

$$W(z) = t_0 + \sum_{j=1}^{n} r_j r^j \cos(j\theta - \theta_j) + i \sum_{j=1}^{\infty} r_j r^j \sin(j\theta - \theta_j) \,, \tag{3.1}$$

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and the behavior of $|e^{-NW(z)}| = e^{-NReW(z)}$ for $r \to \infty$ is dictated by the contribution $r_{j_0} r^{j_0} \cos(j_0\theta - \theta_{j_0})$, where $j_0 = j_0(\theta)$ is the largest j such that $r_j \cos(j\theta - \theta_j) \neq 0$. Namely, $e^{-NW(z)}$ is exponentially decreasing/increasing depending on whether $\cos(j_0\theta - \theta_{j_0})$ is positive/negative. In the very non-generic case when $r_j \cos(j\theta - \theta_j)$ vanishes for all $j = 1, \ldots, n$, the quantity $e^{-NW(z)}$ oscillates indefinitely as $r \to \infty$.

Let us fix θ such that $\cos(n\theta - \theta_n) \neq 0$. Then $e^{-NW(z)}$ is exponentially decreasing as $r \to \infty$ if and only if $\cos(n\theta - \theta_n) > 0$, which gives:

$$\theta = \frac{\alpha + \theta_n}{n} + \pi\frac{2k}{n} \qquad \text{with } k = 0, \ldots, n-1 \quad \text{and} \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.2}$$

This defines n angular sectors (=open cones with apex at the origin) in the complex plane, which we denote by A_k, $k = 0, \ldots, n-1$. We also define complementary sectors B_k through:

$$\theta = \frac{\alpha + \theta_n}{n} + \pi\frac{2k+1}{n} \qquad \text{with } k = 0, \ldots, n-1 \quad \text{and} \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \tag{3.3}$$

these are the sectors for which $\cos(n\theta - \theta_n) < 0$. The sectors A_k and B_k arise consecutively with respect to the trigonometric order and are separated by rays originating at $z = 0$.

To make the integral (2.14) absolutely convergent, we shall require that the curve γ asymptotes to some straight lines $d_\pm(t) = \pm\nu_\pm t + \mu_\pm$ for $t \to \pm\infty$, such that the corresponding asymptotes lie inside two of the convergence sectors A_k. That is, we require:

$$\begin{aligned} &\exists \lim_{t \to \pm\infty} (\gamma(t) \mp \nu_\pm t - \mu_\pm) = 0 \\ &\exists \lim_{t \to \pm\infty} \gamma'(t) := \pm\nu_\pm \,. \end{aligned} \tag{3.4}$$

for some $\nu_\pm \in A_{k_\pm}$ and some $\mu_\pm \in \mathbb{C}$. Here $\gamma'(t) = d\gamma(t)/dt$ and $k_\pm \in \{0, \ldots, n-1\}$. Since the integrand of (2.14) is holomorphic, it immediately follows that the partition function is independent of the choice of γ as long as this contour has asymptotic behavior (3.4) with $\nu_\pm$ belonging to fixed sectors $A_{k_\pm}$. In particular, the integral does not depend on the precise values of $\nu_\pm$ and $\mu_\pm$, but only on the convergence sectors A_{k_+} and A_{k_-}. Therefore, we obtain n^2 possible values $\mathcal{Z}(k_-, k_+, t)$, indexed by the sectors A_{k_-} and A_{k_+} (figure 1). A complete definition of the model requires that we specify one of the 'phases' (k_-, k_+), together with the potential W.

It is also clear that $\mathcal{Z}(k_-, k_+, t)$ vanishes if $k_+ = k_-$, and that we have the relation:

$$\mathcal{Z}(k_-, k_+, t) = (-1)^N \mathcal{Z}(k_+, k_-, t)\,, \tag{3.5}$$

which results upon reversing the orientation of γ. Therefore, it suffices to consider the $n(n-1)/2$ 'phases' indexed by pairs (k_-, k_+) with $k_- > k_+$.

3.1 Scaling symmetry

Let q be a non-vanishing complex number. Writing $z := x/q$, we have:

$$W(z) = W_q(x)\,, \tag{3.6}$$

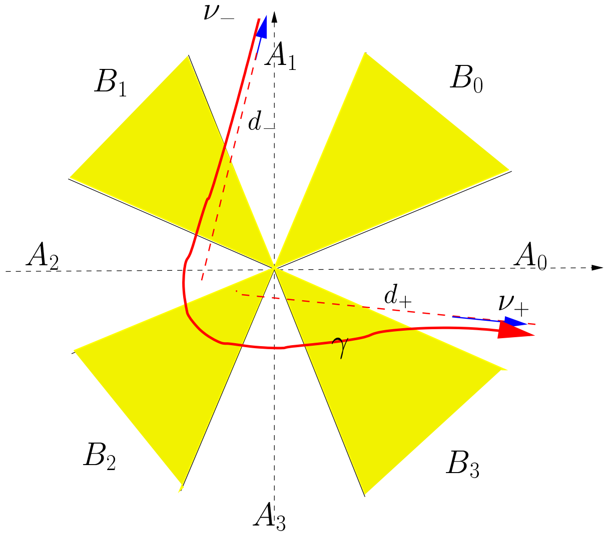

Figure 1: Convergence sectors of the holomorphic matrix model. We show the case $n = degW = 4$, with $\theta_4 = 0$ and a contour belonging to the sector $(k_-, k_+) = (1, 0)$.

where $W_q(x) = \sum_{j=0}^{n} t_j^{(q)} x^j$, with $t_j^{(q)} := t_j/q^j$. Performing the change of variables $\lambda := \mu/q$ in (2.14) gives:

$$\mathcal{Z}_N(\gamma, t) = \frac{1}{q^{N^2}} \mathcal{Z}_N(\gamma_q, t^{(q)}) \,, \tag{3.7}$$

where γ_q is the path defined through:

$$\gamma_q(t) = q\gamma(t) \tag{3.8}$$

for all real t.

For the choice $q_n := r_n^{1/n} e^{-\frac{i\theta_n}{n}}$, the change of variable $z := x/q_n$ gives $t_n^{(q_n)} = 1$, so the transformed potential W_{q_n} has unit leading coefficient. Hence the model depends 'trivially' on the parameter t_n, and we can set $t_n = 1$ and $\theta_n = 0$ without loss of generality.

Also note that choosing $q := \alpha^k$ with $\alpha = e^{\frac{2\pi i}{n}}$ and k an integer allows one to rotate γ by any multiple of $2\pi/n$. Since this does not change the convergence sectors (because $t_n^{(\alpha^k)} = t_n$), it leads to the relation:

$$\mathcal{Z}_N(k + k_-, k + k_+, \{t_m\}) = \alpha^{N^2 k} \mathcal{Z}_N(k_-, k_+, \{\alpha^{mk} t_m\}) \,. \tag{3.9}$$

Hence it suffices to consider the 'phases' $(k_-, 0)$ with $k_- = 1, \ldots, n-1$.

Observation. If one increases the degree of W, then the convergence sectors become progressively narrower. Allowing for the case $n = \infty$ (i.e. for an entire function W) is often a useful device in the theory of matrix models (the best known example is the matrix model of Kontsevich [24] and its generalizations). In this case, the convergence structure of the holomorphic model can be quite different from that discussed above, and depends on the precise asymptotic behavior of the entire function W. A simple example is provided by the choice $W(z) = e^z$, which gives $|e^{-NW(z)}| = e^{-NRe\ W(z)} = e^{-Ne^x \cos(y)}$, where $z = x + iy$

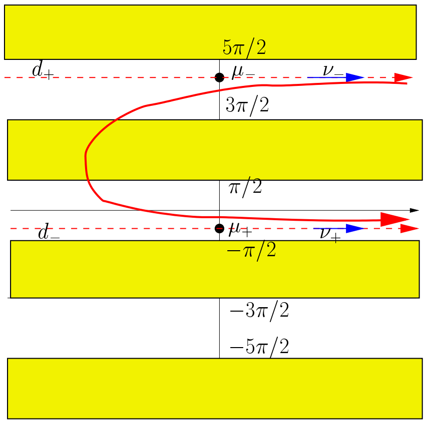

Figure 2: Convergence sectors for an entire potential $W = e^z$, and a good choice of contour for such a model. The filled-in regions are forbidden sectors for $\mu_\pm$.

with x, y real. Then the convergence sectors are horizontal strips defined by the condition:

$$\cos(y) > 0 \Longleftrightarrow y \in \left(-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right), \qquad k \in \mathbb{Z}\,. \tag{3.10}$$

Correspondingly, we obtain a convergent partition function by taking γ to satisfy (3.4), where now $\nu_\pm > 0$ and $\mu_\pm$ belong to two such bands (figure 2). This example should serve as warning against the idea that one can recover models with power series potentials as naive limits of polynomial models.

3.2 Even and odd degree potentials, and the hermitean matrix model

3.2.1 The case $n =$ even

In the case $n =$ even, the convergence sectors of the model appear in pairs A_k, $A_{k+n/2}$ which are symmetric with respect to the inversion $z \to -z$. By the discussion above, we can take t_n positive without loss of generality. Then $\theta_n = 0$ and the two sectors A_0 and $A_{n/2}$ lies opposite to each other and are bisected by the positive real axis. Picking $k_- = n/2$ and $k_+ = 0$, we can describe the 'phase' $\mathcal{Z}(n/2, 0, t)$ by taking the curve γ to coincide with the real line (with its usual orientation). Then the partition function (2.14) reduces to the usual eigenvalue representation of the hermitean one-matrix model. This gives a partial justification for the formulation used in [1].

3.2.2 The case $n =$ odd

In this case, one has an odd number of convergence sectors A_k, whose images under the reflection $z \to -z$ are the 'bad' sectors $B_{k+[n/2]}$. Picking t_n to be positive as above, it is clear that the matrix model *cannot* be defined by choosing $\gamma = \mathbb{R}$, since the associated integral would diverge; this amounts to the basic observation that the hermitean matrix model is ill-defined for generic complex polynomial potentials of odd degree. However, one can certainly make sense of the holomorphic model by working with any of the good 'phases' (k_-, k_+) — one simply picks a contour γ such that ν_+ and ν_- belong to some of the convergence sectors A_k.

The fact that the hermitean model cannot be relevant for odd degree potentials is related to problems observed in [8], which in that paper were avoided by declaring that certain matrices occurring from a perturbative analysis should be *anti-hermitean*. A systematic formulation of this idea is to consider the holomorphic matrix model instead. To substantiate this proposal, appendix B shows that the procedure of [8] admits a simple justification in the framework of holomorphic models.

4. Loop equations, equations of motion and the large-N limit

In this section, we study the loop equations and equations of motion of the holomorphic model, as well as its planar limit. As we shall see below, much of the standard fare of hermitean models can be extended quite directly to the holomorphic case (though there are a few modifications). The most important novel feature of holomorphic matrix models is that they can explore an enlarged set of planar limits, thus probing the entire moduli space of a certain family of algebraic curves. This fact, which is essential for the conjecture of [1], will be established explicitly by proving a reconstruction theorem which associates a planar solution of the model with an arbitrary algebraic curve belonging to this family.

Let ds be the length element on γ and s the length coordinate along this curve, centered at some point on γ. We let $\lambda(s)$ denote the parameterization of γ with respect to this coordinate and write $\lambda_i = \lambda(s_i)$ accordingly. Mimicking usual constructions, we define the spectral density by:

$$\rho(s) = \frac{1}{N}\sum_{j=1}^{N}\delta(s - s_j)\,, \tag{4.1}$$

where $\delta(s - s_j)$ is the delta-function in the coordinate s (equivalently, the δ-function along γ with respect to the measure induced by the length element). Note the normalization condition:

$$\int_{-\infty}^{\infty} ds\rho(s) = 1\,. \tag{4.2}$$

Also consider the resolvent of M:

$$R(z) := \frac{1}{z - M}\,, \tag{4.3}$$

and its normalized trace:

$$\omega(z) = \frac{1}{N}\,\mathrm{tr}\,R(z) = \frac{1}{N}\,\mathrm{tr}\,\frac{1}{z - M} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z - \lambda_i} = \int ds\frac{\rho(s)}{z - \lambda(s)}\,. \tag{4.4}$$

In the following, we shall need the Sokhotsky-Plemelj formulae:

$$\lim_{\epsilon\to 0^+}\int_\gamma d\lambda'\frac{1}{\lambda - \lambda' \pm \epsilon n(\lambda)} = \mathcal{P}\int_\gamma d\lambda'\frac{1}{\lambda - \lambda'} \mp i\pi \text{for } \lambda \in \gamma\,, \tag{4.5}$$

which we also write symbolically as:

$$\lim_{\epsilon\to 0^+}\frac{1}{\lambda(s) - \lambda(s') \pm \epsilon n(s)} = \mathcal{P}\frac{1}{\lambda(s) - \lambda(s')} \mp \frac{i\pi}{\dot\lambda(s)}\delta(s - s')\,. \tag{4.6}$$

Here $\mathcal{P}$ stands for the principal value and $n(s) = i\dot\lambda(s)$ is the normal vector field to γ (figure 3).

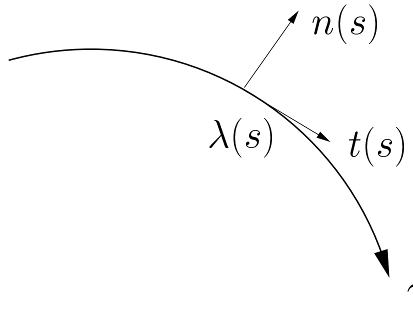

Figure 3: The normal vector field $n(s) = it(s)$, and the tangent vector field $t(s) = \dot{\lambda}(s)$ of γ. Note that $|n(s)| = |t(s)| = 1$ since s is the length coordinate along γ.

4.1 Loop equations

To extract the loop equations, we follow the method of [25]. For this, start with the identity:

$$\int_\gamma d\lambda_1 \cdots \int_\gamma d\lambda_N \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \left[\prod_{k \neq l} (\lambda_k - \lambda_l) e^{-N \sum_{j=1}^{N} W(\lambda_j)} \frac{1}{z - \lambda_i} \right] = 0\,. \tag{4.7}$$

Performing the partial differentiation, we write this as:

$$\left\langle \sum_{i=1}^{N} \frac{1}{(z-\lambda_i)^2} - N \sum_{i=1}^{N} \frac{W'(\lambda_i)}{z - \lambda_i} + 2 \sum_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)(z - \lambda_i)} \right\rangle = 0\,. \tag{4.8}$$

Using the decomposition:

$$\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha - \beta} \left[\frac{1}{z-\alpha} - \frac{1}{z-\beta} \right] \tag{4.9}$$

to simplify the last term and combining the result with the first gives:

$$\left\langle \omega(z)^2 - \frac{1}{N} \sum_{i=1}^{N} \frac{W'(\lambda_i)}{z - \lambda_i} \right\rangle = 0\,. \tag{4.10}$$

Defining the polynomial:

$$f(z) := \frac{1}{N} \sum_{i=1}^{N} \frac{W'(z) - W'(\lambda_i)}{z - \lambda_i} = \int ds \rho(s) \frac{W'(z) - W'(\lambda(s))}{z - \lambda(s)}\,, \tag{4.11}$$

relation (4.10) gives the following form of the loop equations:

$$\langle \omega(z)^2 \rangle - W'(z) \langle \omega(z) \rangle + \langle f(z) \rangle = 0\,. \tag{4.12}$$

4.2 Equations of motion

Writing $\lambda_i = \lambda(s_i)$, the partition function (2.14) becomes:

$$\begin{aligned}\mathcal{Z}_N(\gamma,t) &= \int ds_1 \cdots \int ds_N \prod_{i=1}^N \dot\lambda(s_i) \prod_{i\neq j} (\lambda(s_i)-\lambda(s_j)) e^{-N\sum_{j=1}^N W(\lambda(s_i))} \\ &= \int ds_1 \cdots \int ds_N e^{-N S_{\rm eff}(s_1 \dots s_N)}\,, \end{aligned} \tag{4.13}$$

where $\dot\lambda(s) := d\lambda(s)/ds$ and:

$$S_{\rm eff}(s_1 \dots s_N) = \sum_{j=1}^N W(\lambda(s_i)) - \frac{1}{N}\sum_{i\neq j}\ln(\lambda_i - \lambda_j) - \frac{1}{N}\sum_{i=1}^N \ln\dot\lambda_i\,. \tag{4.14}$$

Extremizing with respect to s_i gives the equations of motion:

$$\frac{2}{N}{\sum_j}' \frac{1}{\lambda(s_i)-\lambda(s_j)} = W'(\lambda(s_i)) - \frac{1}{N}\frac{\ddot\lambda(s_i)}{\dot\lambda(s_i)^2}\,. \tag{4.15}$$

The prime means that the sum is taken only over $j \neq i$. The last term is a curvature-induced contribution which is subleading in $1/N$. It is also easy to check that the equations of motion imply:

$$\omega(z)^2 - W'(z)\omega(z) + f(z) + \frac{1}{N}\frac{d}{dz}\omega(z) + \frac{1}{N^2}\sum_{i=1}^N \frac{\ddot\lambda(s_i)}{\dot\lambda(s_i)^2} = 0\,. \tag{4.16}$$

This 'operator' equation holds only for solutions of (4.15), unlike the Ward identity (4.12).

4.3 The large-N limit

For any quantity ϕ, consider the large-N expansion:

$$\langle\phi\rangle = \sum_{j=0}^\infty \frac{\phi_j}{N^j}\,, \tag{4.17}$$

with coefficients ϕ_0, ϕ_1 etc. In particular, we have $\langle\rho(s)\rangle = \rho_0(s) + O(1/N)$ and $\langle\omega(z)\rangle = \omega_0(z) + O(1/N)$. In the large-$N$ limit, the eigenvalues λ_j are replaced by the planar spectral density $\rho_0(\lambda)$ supported on the curve γ. Note that this quantity is *complex*, since its definition involves averaging with respect to the complex integrand of (2.14). As usual, we have

$$\omega_0(z) = \int ds \frac{\rho_0(s)}{z-\lambda(s)}\,, \tag{4.18}$$

so that ω_0 becomes an analytic function whose cuts I_α are forced to lie on the curve γ (figure 4). We shall assume for simplicity that all cuts are of finite length.

In the planar limit, the average of a product can be replaced by the product of averages and the loop equations (4.12) reduce to the algebraic constraint:

$$\omega_0(z)^2 - W'(z)\omega_0(z) + f_0(z) = 0\,, \tag{4.19}$$

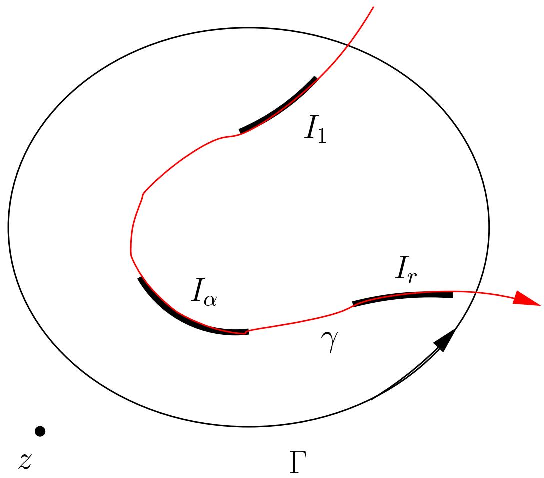

Figure 4: Cuts of ω_0. We also show a closed contour Γ surrounding all cuts and a point z in its exterior.

where

$$f_0(z) = \int ds \rho_0(s) \frac{W'(z) - W'(\lambda(s))}{z - \lambda(s)} \tag{4.20}$$

is a polynomial of degree $n-2$ with complex coefficients. Since $\rho(s)$ is normalized by (4.2), equation (4.20) shows that the leading coefficient of $f_0(z)$ equals nt_n.

In the planar limit, the equations of motion (4.15) become:

$$2\mathcal{P} \int ds' \frac{\rho_0(s')}{\lambda(s) - \lambda(s')} = W'(\lambda(s)) \,. \tag{4.21}$$

The curvature term in (4.15) drops out, since it is subleading in $1/N$. The algebraic constraint (4.19) can also be obtained from (4.21) by standard manipulations. Writing:

$$\omega_0(z) = u_0(z) + \frac{1}{2} W'(z) \,, \tag{4.22}$$

relation (4.19) shows that $u_0(z)$ is one of the branches of the planar affine curve:

$$u^2 - \frac{1}{4} W'(z)^2 + f_0(z) = 0 \,. \tag{4.23}$$

Since the branch cuts of ω_0 (and thus of u_0) must lie along γ, it is clear that the polynomial f_0 in equation (4.23) is constrained by the choice of this curve. However, changing γ without changing its asymptotes allows one to describe any position of the cuts in the complex plane, as long as all these cuts have finite length. This effectively eliminates the constraints that would be present in the hermitean case (for which γ would be forced to coincide with the real axis).

4.4 Reconstruction of a planar solution from the Riemann surface

Given the algebraic curve (4.23), we now show how one can use it to recover an appropriate γ supporting a spectral density ρ_0 satisfying (4.2), (4.18), (4.20) and the planar equations of motion (4.21). This proves that the holomorphic matrix model is free to explore the whole relevant piece of the moduli space of (4.23), unlike the hermitean matrix model. In the planar limit, the holomorphic model reduces to the singular integral equation (4.21). Up to some minor details, the existence of a one to one relation between solutions of this equation and members of a family of Riemann surfaces boils down to the well-known relation between the Riemann problem and singular integral equations.

To see this explicitly, assume that one is given a complex degree $n-2$ polynomial $f_0(z)$, subject only to the constraint that its leading coefficient equals nt_n. Consider the associated curve (4.23). Denoting its branch points by a_α, b_α (with $\alpha = 1, \ldots, n-1$), choose branch-cuts I_α connecting a_α and b_α (note that we allow I_α to be curved). This defines two determinations, which we call u_0 and $u_1 = -u_0$. More precisely, u_0 is the determination which behaves as $-\frac{1}{2}W'(z)$ for large z. Let us define ω_0 by relation (4.22) and choose a curve γ such that $I_\alpha \subset \gamma$ for all α. We let s be its length coordinate and $\lambda = \lambda(s)$ the associated parameterization. Choosing the normal $n(s) = i\dot{\lambda}(s)$ (figure 3) we define:

$$\rho_0(s) := \dot{\lambda}(s) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left[\omega_0(\lambda(s) - \epsilon n(s)) - \omega_0(\lambda(s) + \epsilon n(s))\right] \tag{4.24}$$

for every s in γ. Then ρ vanishes outside of I_α and (4.24) and the Sokhotsky formulae (4.5) imply:

$$\int ds \frac{\rho_0(s)}{z - \lambda(s)} = \int_\Gamma \frac{dx}{2\pi i} \frac{\omega_0(x)}{z - x} = \omega_0(z)\,, \tag{4.25}$$

where Γ is a contour surrounding all cuts but not the point z (see figure 4) and the last equality follows by deforming this contour toward infinity to pick the contribution from the residue at $x = z$. This shows that relation (4.18) holds. Using (4.23) and the fact that the leading coefficient of f_0 equals nt_n shows that $\omega(z) = 1/z + O(1/z^2)$ for large $|z|$, and combining this with (4.18) shows that ρ_0 satisfies the normalization condition (4.2).

Since the cuts I_α connect the branches u_0 and $-u_0$, we have:

$$\lim_{\epsilon \to 0^+} u_0(\lambda + \epsilon n) = \lim_{\epsilon \to 0^+} \left[-u_0(\lambda - \epsilon n)\right], \qquad \lambda \in I_\alpha \tag{4.26}$$

so that $\lim_{\epsilon \to 0^+} \left[\omega_0(\lambda + \epsilon n) + \omega_0(\lambda - \epsilon n)\right] = W'(\lambda)$ for $\lambda \in I_\alpha$. Combining this with equation (4.18) and using the Sokhotsky formulae (4.5) shows that ρ_0 satisfies the planar equations of motion (4.21) along the cuts.

To prove (4.20), we use relation (4.18) to compute:

$$\omega_0(z)^2 = \int ds \int ds' \frac{\rho_0(s)\rho_0(s')}{(z - \lambda(s))(z - \lambda(s'))} = 2 \int ds \int ds' \frac{\rho_0(s)\rho_0(s')}{(\lambda(s) - \lambda(s'))(z - \lambda(s))}\,, \tag{4.27}$$

where we used the identity (4.9) and symmetry of the resulting integrand with respect to the substitution $s \leftrightarrow s'$. Using (4.21) in the right hand side of (4.27) gives:

$$\omega_0(z)^2 = \int ds \rho_0(s) \frac{W'(\lambda(s))}{z - \lambda(s)} = -\int ds \rho_0(s) \frac{W'(z) - W'(\lambda(s))}{z - \lambda(s)} + W'(z)\omega_0(z)\,. \tag{4.28}$$

Comparing with (4.23) and (4.22) shows that equation (4.20) holds.

This construction produces a planar solution of the holomorphic model which recovers *any* curve of the form (4.23), for an arbitrary choice of degree $n-2$ polynomial $f_0(z)$ with leading coefficient nt_n. Unlike the hermitean model, the holomorphic model explores the entire family of curves (4.23) in its planar limit. This is a basic pre-requisite for the conjecture of [1].

It is also clear that the precise choice of cuts I_α is irrelevant, as long as they connect the given pairs of branch points a_α, b_α. Thus one can use any[2] curve γ in this construction, provided that it passes through all the branch points of (4.23).

5. The microcanonical ensemble

The framework of [1] requires that the model obeys certain filling fraction constraints. In this section, I explain how one can impose such constraints on the *finite* N model.[3] The relevant conditions are easiest to formulate by employing a microcanonical ensemble. As we shall see below, the original path integral defines a (grand-)canonical ensemble at zero chemical potentials. This allows one to recover the microcanonical generating function by introducing chemical potentials (which are canonically conjugate to the filling fractions) and then performing a Legendre transform to replace the former by the latter.

5.1 The (grand-)canonical partition function associated with a collection of domains

We start by fixing a collection of domains D_α ($\alpha = 1, \dots, r$) in the complex plane, chosen such that their interiors are mutually disjoint and such that:

$$\cup_{\alpha=1}^{r} \overline{D}_\alpha = \mathbb{C}. \tag{5.1}$$

We shall assume that γ intersects each closure $\bar{D}_\alpha$ along a single curve segment Δ_α, where Δ_α lie in ascending order on γ with respect to its orientation (figure 5). Condition (5.1) implies:

$$\cup_{\alpha=1}^{r} \Delta_\alpha = \gamma\,. \tag{5.2}$$

For simplicity, we shall take D_α to be infinite strips arranged as in figure 5.

Letting χ_α denote the characteristic function of D_α, consider the matrices $\chi_\alpha(M)$ defined by holomorphic functional calculus:

$$\chi_\alpha(M) = \oint_{\Gamma_\alpha} \frac{dz}{2\pi i} \frac{1}{z-M}\,, \tag{5.3}$$

[2] Nondegenerate and without self-intersections.

[3] A clear formulation of such constraints beyond the strict large-N limit is important in studies of SO(N) and Sp(N) matrix models, whose large-N expansion contains terms of order $1/N$ which must be interpreted as contributions to the dual field theory (rather than gravitational contributions, which start at order $1/N^2$). The formulation given below allows one to give clear proofs of relations between the strictly planar and $O(1/N)$ contributions to the (microcanonical) partition function, thereby strengthening arguments like those given in [11]. Similar problems are encountered in more general orientifold models [26].

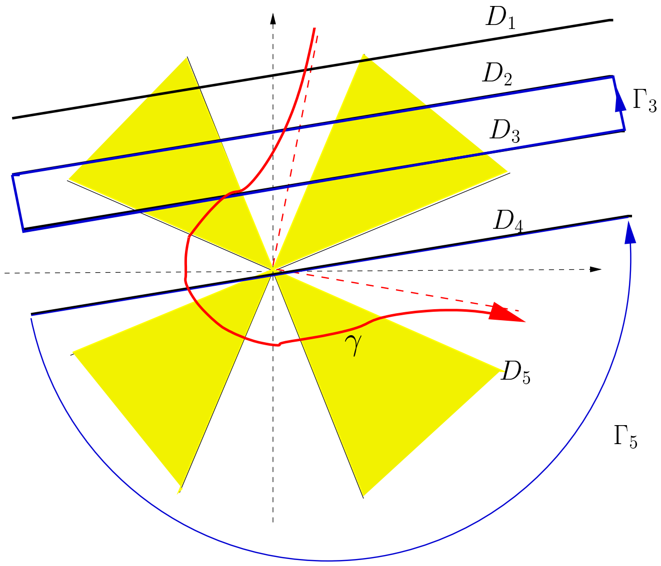

Figure 5: A choice of strip domains in the complex plane. We also show two of the bounding contours (namely Γ_3 and Γ_5). In this example, we take the domains to be infinite strips; this assures us that γ and its deformations will cut each such domain along a *non-void* interval.

with Γ_α a (counterclockwise) contour bounding D_α in the complex plane (figure 5). The matrix $\chi_\alpha(M)$ equals the projector on the space spanned by those eigenvectors of M whose eigenvalues lie in D_α.

Let $f_\alpha := \frac{1}{N}\,\mathrm{tr}\,\chi_\alpha(M) = \frac{1}{N}\sum_{j=1}^N \chi_\alpha(\lambda_j)$ be the filling fractions for the domains D_α. One has:

$$f_\alpha = \int ds\rho(s)\chi_\alpha(\lambda(s)) = \oint_{\Gamma_\alpha} \frac{dz}{2\pi i}\omega(z)\,, \tag{5.4}$$

since $\omega(z)$ has simple poles with residue $\frac{1}{N}$ at each eigenvalue λ_j. Relation (5.2) implies $\sum_{\alpha=1}^r \chi_\alpha(\lambda) = 1$, so the filling fractions are subject to the constraint:

$$\sum_{\alpha=1}^r f_\alpha = 1\,. \tag{5.5}$$

Picking *complex* chemical potentials μ_α, we consider the (grand-)canonical ensemble associated with our collection of domains:

$$\begin{aligned}\mathcal{Z}_N(\gamma;t,\mu) &= \frac{1}{N}\int_{\Gamma(\gamma)} dM e^{\mathrm{tr}\left[-NW(M)-N\sum_{\alpha=1}^r \mu_\alpha\chi_\alpha(M)\right]} \\ &= \int_\gamma d\lambda_1\cdots\int_\gamma d\lambda_N \prod_{i\neq j}(\lambda_i-\lambda_j)e^{-N\sum_{j=1}^N W(\lambda_j)-N^2\sum_{\alpha=1}^r \mu_\alpha f_\alpha}\,,\end{aligned} \tag{5.6}$$

which is an analytic function of μ_α. The original partition function results by setting $\mu_\alpha = 0$, and thus it corresponds to a *(grand-)canonical* ensemble at zero chemical potentials. Notice that the (grand-)canonical partition function can be written:

$$\mathcal{Z}_N(\gamma;t,\mu) = \sum_{N_1+\cdots+N_r=N}\frac{N!}{N_1!\ldots N_r!}e^{-N\sum_{\alpha=1}^N \mu_\alpha N_\alpha}\mathcal{Z}_{N_1\ldots N_r}(\gamma;t)\,, \tag{5.7}$$

where:

$$\mathcal{Z}_{N_1\dots N_r}(\gamma;t) = \int_\gamma d\lambda_1 \cdots \int_\gamma d\lambda_{N_1} \cdots \int_\gamma d\lambda_{N_1+\dots+N_{r-1}+1} \cdots \int_\gamma d\lambda_N \times \\ \times \prod_{i\neq j} (\lambda_i - \lambda_j) e^{-N\sum_{j=1}^N W(\lambda_j)} . \tag{5.8}$$

Observation. One can consider a more general version of microcanonical ensemble based on a finite partition of unity, i.e. a finite collection of smooth complex-valued functions $\phi_\alpha(z,\overline{z})$ satisfying the constraint:

$$\sum_\alpha \phi_\alpha = 1 . \tag{5.9}$$

In certain ways, this is preferable to the approach taken above, since it may be technically desirable to avoid having to deal with characteristic functions. The entire discussion of this section extends easily to this more general set-up.

Introducing the (grand-)canonical generating function:

$$\mathcal{F}_N(\gamma;t,\mu) := -\frac{1}{N^2} \ln \mathcal{Z}_N(\gamma;t,\mu) , \tag{5.10}$$

we have the standard relation:

$$\frac{\partial}{\partial \mu_\alpha}\mathcal{F} = \langle f_\alpha \rangle . \tag{5.11}$$

Here and below, the brackets $\langle \cdots \rangle$ denote the expectation value taken in the *(grand-)canonical* ensemble.

5.2 The microcanonical generating function

Following standard statistical mechanics procedure, we define:

$$S_\alpha := \frac{\partial}{\partial \mu_\alpha}\mathcal{F} \tag{5.12}$$

and perform a Legendre transform to extract the microcanonical generating function:

$$F(\gamma,t,S) := S_\alpha \mu_\alpha(\gamma,t,S) - \mathcal{F}(\gamma,t,\mu_\alpha(t,S)) , \tag{5.13}$$

which is an analytic function of the complex variables t_α and S_α. In this relation, μ_α are expressed in terms of t and S by solving equations (5.12). The constraint (5.5) and equation (5.11) show that S_α satisfy:

$$\sum_{\alpha=1}^N S_\alpha = 1 , \tag{5.14}$$

so we can take $S_1 \dots S_{r-1}$ to be the independent variables. Then equations (5.12) express μ_α as functions of t_j and these coordinates, and equation (5.13) implies:

$$\mu_\alpha - \mu_r = \frac{\partial F}{\partial S_\alpha} \qquad \text{for } \alpha = 1, \dots, r-1 . \tag{5.15}$$

This gives the chemical potentials as functions of t and S_α. Note that μ_α are only determined up to a common constant shift; this is due to the constraint (5.14) on S_α. Working with $F(\gamma, t, S)$ amounts to fixing the expectation values of the filling fractions by imposing the *quantum* constraint (5.12):

$$\langle f_\alpha \rangle = \oint_{\gamma_\alpha} \frac{dz}{2\pi i} \langle \omega(z) \rangle = S_\alpha \,, \tag{5.16}$$

with S_α treated as fixed parameters. We stress that this condition is only imposed on the expectation values of the filling fractions.

5.3 Chemical potentials at large N

We will now show that the large-N chemical potentials can be expressed as B-type periods of the algebraic curve (4.23), thereby proving that the special geometry relations of [1] hold at the holomorphic matrix level. The argument below is an adaptation of that given in [20],[4] combined with the definition of the microcanonical ensemble given above. In particular, we show that the special geometry relations are simply the large-N limit of the standard equations (5.15). Hence the chemical potentials μ_α are the 'quantum' (i.e. finite N) analogues of the B-type periods of [1], while the averaged filling fractions are the 'quantum' analogues of the A-type periods. This captures the beautiful intuition of [1].

For this, we start from the expression of the (grand-)canonical generating function in the planar limit:

$$\mathcal{F}_0(\gamma, t, \mu) = \int ds W(\lambda(s))\rho_0(s) - \mathcal{P} \int ds \int ds' \tilde{K}(\lambda(s), \lambda(s'))\rho_0(s)\rho_0(s') + \sum_{\alpha=1}^{r} \mu_\alpha \int_{I_\alpha} ds \rho_0(s) \tag{5.17}$$

where:

$$\tilde{K}(\lambda, \lambda') := \ln(\lambda - \lambda') \,. \tag{5.18}$$

In what follows, we shall assume that each cut I_α lies inside a corresponding domain D_α; in particular, we assume that the number of cuts coincides with the number of domains. With this assumption, we have $S_\alpha = \langle f_\alpha \rangle = \int_{I_\alpha} ds \rho_0(s)$ and the Legendre transform of (5.17) gives the planar limit of the microcanonical generating function:

$$F_0(\gamma, t, S) = \mathcal{P} \int ds \int ds' \tilde{K}(\lambda(s), \lambda(s'))\rho_0(s)\rho_0(s') - \int ds W(\lambda(s))\rho_0(s) \,, \tag{5.19}$$

with the constraints:

$$\int_{I_\alpha} ds \rho_0(s) = S_\alpha \qquad \text{for } \alpha = 1, \ldots, r \,. \tag{5.20}$$

Remember that we allow I_α to be curved intervals connecting the branch points a_α and b_α of the algebraic curve (4.23). In the generic case, none of the cuts is reduced to a double point and one can take $r = n - 1$.

---

[4] In [1], Dijkgraaf and Vafa gave a beautiful intuitive justification for the special geometry relation $\Pi_\alpha = \partial F_0 / \partial S_\alpha + \text{const}$, where F_0 is the planar limit of the (microcanonical) generating function while S_α and Π_α are identified with periods of the curve (4.23). A derivation of this relation (in the context of the hermitean model) was later given in [20], upon using older results of [27].

5.3.1 The primitive of u_0 along γ

To extract the large-N chemical potentials, consider the 'restriction' of u_0 along γ, which we define by:

$$u_0^p(s) := \frac{1}{2} \lim_{\epsilon \to 0^+} \left[u_0(\lambda(s) + \epsilon n(s)) + u_0(\lambda(s) - \epsilon n(s)) \right]. \tag{5.21}$$

Here $n(s) = i\dot{\lambda}(s)$.

If $\lambda(s)$ is a point of γ lying outside the union of I_α, then $u_0^p(s)$ equals $u_0(\lambda(s))$, the quantity obtained by substituting $\lambda(s)$ for z in the expression:

$$u_0(z) = \omega_0(z) - \frac{1}{2}W'(z) = \int ds' \frac{\rho_0(s')}{z - \lambda(s')} - \frac{1}{2}W'(z). \tag{5.22}$$

Using (5.22) in the definition of u_0^p gives:

$$u_0^p(s) := \mathcal{P} \int ds' \rho_0(s') K(\lambda(s), \lambda(s')) - \frac{1}{2} W'(\lambda(s)), \tag{5.23}$$

where:

$$K(\lambda, \lambda') = \frac{1}{\lambda - \lambda'} \tag{5.24}$$

is the integral kernel appearing in the planar loop equations and where we used the Sokhotsky identities (4.5).

Consider now the function $\phi : \gamma \to \mathbb{C}$ defined though:

$$\phi(s) := 2\mathcal{P} \int ds' \tilde{K}(\lambda(s), \lambda(s')) \rho_0(s') - W(\lambda(s)). \tag{5.25}$$

Noticing that $K(\lambda, \lambda') = \frac{\partial}{\partial \lambda} \tilde{K}(\lambda, \lambda')$, we have:

$$\frac{d}{ds}\phi(s) = 2\dot{\lambda}(s) u_0^p(s). \tag{5.26}$$

As clear from (5.23), the planar equations of motion (4.21) amount to the requirement that u_0^p vanishes along each of the curve segments I_α:

$$u_0^p(s) = 0 \qquad \text{for } \lambda(s) \in I := \cup_{\alpha=1}^{r} I_\alpha. \tag{5.27}$$

This means that ϕ is constant along each of these intervals:

$$\phi(s) = \xi_\alpha = \text{constant on } I_\alpha, \tag{5.28}$$

The jump in the value of u_0 between consecutive cuts can be obtained by integrating (5.26):

$$\xi_{\alpha+1} - \xi_\alpha = 2 \int_{b_\alpha}^{a_{\alpha+1}} d\lambda u_0(\lambda), \tag{5.29}$$

where we used $d\lambda = \dot{\lambda}(s) ds$. This integral is of course taken along γ.

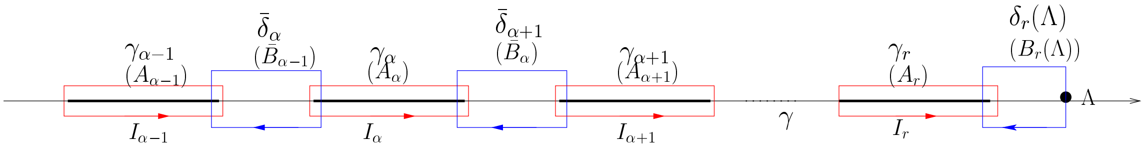

Figure 6: Choice of A and B-cycles on the large-N Riemann surface (for convenience, we represent the curve γ as a straight line, though this need not be the case). The figure indicates the projections γ_α, $\bar{\delta}_\alpha$ and $\delta_r(\Lambda)$ of the cycles A_α, $\bar{B}_\alpha$ and B_r onto the z plane. The cycles A_α can be identified with their projections γ_α. The cycles $\bar{B}_\alpha$ are defined such that, when crossing the cut I_α going upwards along these cycles, one moves from the branch u_0 to the branch $u_1 = -u_0$. Thus the 'lower halves' of these cycles lie on the branch u_0, while their 'upper halves' lie on the branch u_1. A similar convention is used for $B_r(\Lambda)$. With the orientation of the Riemann surface induced by its complex structure, this implies the intersections $A_\alpha \cap \bar{B}_\alpha = +1$ (note that the cycles A_α and $\bar{B}_\alpha$ intersect in a single point, which lies on the branch u_0).

5.3.2 The planar chemical potentials

Differentiating (5.19) with respect to S_α for some $\alpha < r$ and using relation (5.25) gives:

$$\mu_\alpha^{(0)} - \mu_r^{(0)} = \frac{\partial}{\partial S_\alpha} F_0(\gamma, t, S) = \int_{\cup_{\beta=1}^r I_\beta} ds \frac{\partial \rho_0(s)}{\partial S_\alpha} \phi(s) = \xi_\alpha - \xi_r \,, \tag{5.30}$$

where $\mu_\alpha^{(0)}$ are the planar limits of the chemical potentials. To arrive at the last equality, we used equation (5.28) and noticed that $\int_{I_\beta} \frac{\partial \rho_0(\lambda)}{\partial S_\alpha} = \frac{\partial}{\partial S_\alpha} \int_{I_\beta} ds \rho_0(s)$ equals $\delta_{\alpha\beta}$ if $\beta < r$ and -1 if $\beta = r$ (by virtue of (5.20) and (5.14)). Relation (5.30) shows that the planar chemical potentials coincide with the quantities ξ_α, up to a common additive constant. Using relation (5.29), we obtain:

$$\mu_{\alpha+1}^{(0)} - \mu_\alpha^{(0)} = 2 \int_{b_\alpha}^{a_{\alpha+1}} d\lambda u_0(\lambda) \tag{5.31}$$

As explained above, the quantity $u_0(z)$ has cuts precisely along the intervals I_α. Since the other branch of (4.23) is given by $u_1(z) = -u_0(z)$, this allows us to write (5.31) in the form:

$$\mu_\alpha^{(0)} - \mu_{\alpha+1}^{(0)} = \int_{b_\alpha}^{a_{\alpha+1}} dz[u_1(z) - u_0(z)] = \oint_{\bar{B}_\alpha} dz u(z) \,, \tag{5.32}$$

where $\bar{B}_\alpha$ are cycles on the large-N Riemann surface chosen as explained in figure 6.

Consider the cycles $\tilde{B}_\alpha = \sum_{\beta=\alpha}^{r-1} \bar{B}_\beta$ for all $\alpha = 1, \ldots, r-1$. Then (5.32) implies:

$$\mu_\alpha^{(0)} = \mu_r^{(0)} + \oint_{\tilde{B}_\alpha} dz u(z) \qquad \text{for } \alpha = 1, \ldots, r-1 \,. \tag{5.33}$$

The quantity $\mu_r^{(0)}$ is undetermined and can be fixed arbitrarily. Following [1], we take $\mu_r^{(0)} = \oint_{B_r} dz u(z)$, where Λ is a point close to infinity and $B_r(\Lambda)$ is the cycle described in figure 6. Defining $B_\alpha(\Lambda) = \tilde{B}_\alpha + B_r(\Lambda)$ for all $\alpha = 1, \ldots, r-1$, we obtain:

$$\mu_\alpha^{(0)} = \Pi_\alpha \qquad \text{for } \alpha = 1, \ldots, r \,, \tag{5.34}$$

with:

$$\Pi_\alpha := \int_{B_\alpha} dz u(z) \,. \tag{5.35}$$

Relation (5.34) shows that the chemical potentials μ_α are the finite N analogues of the periods Π_α.

Note also that the filling fractions can be expressed as periods of the meromorphic differential udz over the cycles A_α of figure 6:

$$S_\alpha = \oint_{\gamma_\alpha} \frac{dz}{2\pi i}\omega_0(z) = \oint_{\gamma_\alpha} \frac{dz}{2\pi i}u_0(z) = \oint_{A_\alpha} \frac{dz}{2\pi i}u(z) \,. \tag{5.36}$$

In the second equality, we used relation (4.22) and the fact that $W(z)$ is a polynomial. Equation (5.15) now gives the special geometry relation of [1]:

$$\Pi_\alpha - \Pi_r = \frac{\partial}{\partial S_\alpha}F_0 \,. \tag{5.37}$$

Note that the quantity in the right hand side is the planar limit of the *microcanonical* generating function.

It is clear from figure 6 that $A_\alpha \cap \bar{B}_\alpha = -A_\alpha \cap \bar{B}_{\alpha-1} = +1$ and $A_\alpha \cap \bar{B}_\beta = 0$ if $\beta \neq \alpha, \alpha-1$. This gives $A_\alpha \cap B_\beta = -B_\alpha \cap A_\beta = \delta_{\alpha\beta}$. Since we also have $A_\alpha \cap A_\beta = B_\alpha \cap B_\beta = 0$, it follows that A_α, B_β have canonical intersection form. Hence we have a canonical system of cycles A_α, B_α with $\alpha = 1, \ldots, r$:

$$A_\alpha \cap B_\beta = -B_\alpha \cap A_\beta = \delta_{\alpha,\beta} \,, \quad A_\alpha \cap A_\beta = B_\alpha \cap B_\beta = 0 \qquad \text{for all } \alpha = 1, \ldots, r \,. \tag{5.38}$$

This shows that the properties essential for the conjecture of [1] hold at the level of the holomorphic matrix model.

6. Holomorphic ADE models

In this section, we consider the case of holomorphic ADE models, focusing on the simple example of the holomorphic A_2 model. As mentioned in the introduction, the hermitean approach to such models (and its attending regularization, discussed in [21]) leads to various technical problems which would violate the conjecture of [1]. The purpose of this section is to show explicitly how such issues are avoided in the holomorphic set-up, and to provide a reconstruction theorem similar to the one found for the one-matrix case. In particular, we shall extract explicitly the associated Riemann surface (which is expected from the work of of [28]) and show that one must use a certain 'renormalization' procedure in order eliminate unwanted constraints on its moduli. In fact, we shall find that the curve expected from the work of [28] and [22] can be obtained by using a certain regulator (which is natural in the holomorphic set-up), though only in the limit where this regulator is removed. As we shall see below, this limit can be described as a statistical ensemble of 'reduced' holomorphic models.

6.1 Construction of the models

Consider an ADE quiver diagram with nodes indexed by $\alpha = 1, \ldots, \kappa$ where κ is the rank of the associated simply-laced group. Consider $N_\alpha \times N_\alpha$ complex matrices $\Phi^{(\alpha)}$ for each node, and $N_\alpha \times N_\beta$ complex matrices $Q^{(\alpha\beta)}$ for each pair of nodes α, β which are connected by an edge (in particular, we have *two* matrices $Q^{(\alpha\beta)}$ and $Q^{(\beta\alpha)}$ for each edge of the quiver). We let $s_{\alpha\beta} = s_{\beta\alpha}$ be the incidence matrix of the quiver and $c_{\alpha\beta} = 2\delta_{\alpha\beta} - s_{\alpha\beta}$ be the associated Cartan matrix.

By analogy with [21], we define the *holomorphic ADE matrix model* associated with this quiver by:

$$\mathcal{Z} = \int_\Gamma \prod_{\alpha=1}^{\kappa} d\Phi^{(\alpha)} \prod_{\alpha<\beta} \left[dQ^{(\alpha\beta)} dQ^{(\beta\alpha)}\right] e^{-N\sum_{\alpha=1}^{\rho} W_\alpha(\Phi^{(\alpha)}) + W_{\rm int}(\Phi, Q)}\,, \tag{6.1}$$

where W_α are polynomials of degrees n_α and:

$$W_{\rm int}(\Phi, Q) := \sum_{\alpha<\beta} s_{\alpha\beta} \left[{\rm tr}(Q^{(\alpha\beta)}\Phi^{(\beta)}Q^{(\beta\alpha)}) - {\rm tr}(Q^{(\beta\alpha)}\Phi^{(\alpha)}Q^{(\alpha\beta)})\right]. \tag{6.2}$$

The gauge group is:

$$G := \prod_{\alpha=1}^{\kappa} {\rm GL}(N_\alpha, \mathbb{C})\,, \tag{6.3}$$

with the the obvious action:

$$\left(\Phi^{(\alpha)}, Q^{(\alpha\beta)}\right) \to \left(S_\alpha \Phi^{(\alpha)} S_\alpha^{-1}, S_\alpha Q^{(\alpha\beta)} S_\beta^{-1}\right) \tag{6.4}$$

for $S_\alpha \in {\rm GL}(N_\alpha, \mathbb{C})$.

To completely specify the model, one must choose an appropriate integration manifold Γ. Before explaining our choice, let us comment on the hermitean approach [21]. In that case, one takes Γ to consists of matrices $\Phi^{(\alpha)}, Q^{(\alpha\beta)}$ such that $\Phi^{(\alpha)}$ are hermitean and $Q^{(\beta\alpha)} = (Q^{(\alpha\beta)})^\dagger$ for neighboring nodes α and β. Such a prescription makes sense only if all W_α have even degree (otherwise, the integral diverges because the absolute value of the integrand explodes when the norm of some $\Phi^{(\alpha)}$ is large). However, the hermitean prescription immediately leads to other problems, arising from the integrals over Q. Indeed, it is easy to see that these will bring infinite contributions when some eigenvalue $\lambda_i^{(\alpha)}$ of $\Phi^{(\alpha)}$ coincides with some eigenvalue $\lambda_j^{(\beta)}$ of $\Phi^{(\beta)}$ for neighboring α and β. In the hermitean framework, the solution to this problem is to require that the eigenvalues of neighboring $\Phi^{(\alpha)}$ can never coincide — for example, by taking $\Phi^{(\alpha)}$ to have alternately negative and positive eigenvalues [21]. It turns out that this prescription would violate the conjecture of [1]. To understand why, consider for simplicity the hermitean A_2 model (whose holomorphic version is studied below). The large-N limit of this hermitean model can be extracted by an argument which is formally identical to that presented in appendix C,[5] and is governed by an algebraic curve which is a triple cover of the complex plane. Denoting its three branches

[5]Except that the third equation in (6.31) used in the appendix never plays any role for the regularization of [21].

by u_1, u_2 and u_3 (in a convenient enumeration), one finds that cuts connecting u_1 and u_2 would correspond precisely to loci where *equal* eigenvalues of $\Phi^{(1)}$ and $\Phi^{(2)}$ accumulate — a situation which is forbidden by the regularization prescription used to define the model! Therefore, the regularization prescription of the hermitean model requires that all such cuts are reduced to double points, which means that the large-N curve must always be singular. Moreover, this violates the requirements of the dual field theory, since it would require that some filling fractions vanish identically, thereby violating the study of gaugino condensates performed in [22]. Of course, the nonzero cuts of the resulting curve would also be constrained to alternately lie on the positive and negative halves of the real axis. It should be clear from this discussion that hermitean ADE models are quite unnatural for the conjecture of [1]. Below, we shall show how a certain regularization prescription of the holomorphic A_2 model allows one to obtain an ensemble whose large-N limit satisfies the basic requirements of the conjecture. Unlike the hermitean regularization of [21], the 'holomorphic' regulator used below can be removed in a manner which allows us to recover a smooth Riemann surface.

Returning to the holomorphic model, we shall choose the integration manifold Γ as follows:

1. Fix contours $\gamma^{(\alpha)}$ in the complex plane such that:
$$\gamma^{(\alpha)} \cap \gamma^{(\beta)} = \emptyset \quad \text{for neighboring } \alpha \text{ and } \beta\,. \tag{6.5}$$
and such that each $\gamma^{(\alpha)}$ connects two convergence sectors of W_α (as defined in section 3).

2. Let $\Delta(\gamma)$ be the set of all matrices $(D^{(\alpha)}, Q^{(\alpha\beta)})$ (with $\alpha, \beta = 1, \ldots, \kappa$) which satisfy:
   (a) $D^{(\alpha)} = \mathrm{diag}(\lambda_1^{(\alpha)} \ldots \lambda_{N_\alpha}^{(\alpha)})$, with *distinct* $\lambda_1^{(\alpha)} \ldots \lambda_{N_\alpha}^{(\alpha)}$.
   (b) $\lambda_j^{(\alpha)}$ lies along $\gamma^{(\alpha)}$ for each α and j
   (c) $Q_{ji}^{(\beta\alpha)} = \frac{\overline{Q}_{ij}^{(\alpha\beta)}}{\lambda_i^{(\alpha)} - \lambda_j^{(\beta)}}$ (here the bar denotes complex conjugation).

3. Finally, we let Γ be the union of G-orbits[6] of elements of Δ under the action (6.4). Gauge-fixing the action (6.1) gives the eigenvalue representation:
$$\begin{aligned}\mathcal{Z} = &\int_{\gamma^{(1)}} d\lambda_1^{(1)} \cdots \int_{\gamma^{(1)}} d\lambda_{N_1}^{(1)} \cdots \int_{\gamma^{(\rho)}} d\lambda_1^{(\rho)} \cdots \\ &\cdots \int_{\gamma^{(\rho)}} d\lambda_{N_\rho}^{(\rho)} \frac{\prod_{\alpha, i \neq j} (\lambda_i^{(\alpha)} - \lambda_j^{(\alpha)})}{\prod_{\alpha < \beta, i, j} (\lambda_i^{(\alpha)} - \lambda_j^{(\beta)})^{s_{\alpha\beta}}} e^{-N \sum_{\alpha=1}^{\rho} \sum_{i=1}^{N_\alpha} W_\alpha(\lambda_i^{(\alpha)})}\end{aligned} \tag{6.6}$$
where we dropped constant prefactors and used the identity:
$$\int_\sigma du \wedge dv e^{-(\lambda - \lambda')uv} = -\frac{4\pi i}{\lambda - \lambda'}\,, \tag{6.7}$$

[6] To be more precise, one has to take into account the action of an appropriate group of permutations on the set Δ. This can be done as in appendix A by working with a fundamental domain of Δ under this discrete action.

with $\lambda \neq \lambda'$ complex and σ the contour in $\mathbb{C}^2$ given by:

$$v = \frac{\overline{u}}{\lambda - \lambda'} . \tag{6.8}$$

Condition (6.5) acts as a 'complex' regulator for the holomorphic model, by preventing common eigenvalues of neighboring $\Phi^{(\alpha)}$ and $\Phi^{(\beta)}$. As in the hermitean case, working with the regularized model would therefore not suffice to recover the entire moduli space of planar solutions required by the conjecture of [1]. To eliminate the constraints, the regularization condition (6.5) must be removed by taking the limit of coinciding $\gamma^{(\alpha)}$. This limit can be performed in such a way that the end result is a 'renormalized' model which can be described as a statistical ensemble of 'reduced' holomorphic models. We now show how this works for the case of holomorphic A_2 models.

6.2 Example: the holomorphic A_2 model

For an A_2 quiver, one has $\rho = 2$ and four matrices $\Phi^{(1)}$, $\Phi^{(2)}$, $Q^{(12)}$ and $Q^{(21)}$. The partition function takes the form:

$$\begin{aligned} \mathcal{Z} = &\int_{\gamma^{(1)}} d\lambda_1^{(1)} \cdots \int_{\gamma^{(1)}} d\lambda_{N_1}^{(1)} \int_{\gamma^{(2)}} d\lambda_1^{(2)} \cdots \\ &\cdots \int_{\gamma^{(2)}} d\lambda_{N_2}^{(2)} \frac{\prod_{i\neq j} (\lambda_i^{(1)} - \lambda_j^{(1)}) \prod_{i\neq j} (\lambda_i^{(2)} - \lambda_j^{(2)})}{\prod_{i,j} (\lambda_i^{(1)} - \lambda_j^{(2)})} e^{-N\sum_{i=1}^{N_1} W_1(\lambda_i^{(1)}) - N\sum_{i=1}^{N_2} W_2(\lambda_i^{(2)})} \end{aligned} \tag{6.9}$$

for two disjoint curves $\gamma^{(1)}$ and $\gamma^{(2)}$.

6.2.1 Classical vacua

Extremizing the action:

$$S = N \operatorname{tr}\left[W_1(\Phi^{(1)}) + W_2(\Phi^{(2)})\right] + \operatorname{tr}\left[Q^{(21)}\Phi^{(1)}Q^{(12)} - Q^{(12)}\Phi^{(2)}Q^{(21)}\right] \tag{6.10}$$

gives the equations:

$$\begin{aligned} NW_1'(\Phi^{(1)}) &= -Q^{(12)}Q^{(21)} , & NW_2'(\Phi^{(2)}) &= Q^{(21)}Q^{(12)} , \\ \Phi^{(2)}Q^{(21)} &= Q^{(21)}\Phi^{(1)} , & \Phi^{(1)}Q^{(12)} &= Q^{(12)}\Phi^{(2)} . \end{aligned} \tag{6.11}$$

Combining these, one easily obtains:

$$\begin{aligned} W_1'(\Phi^{(1)}) \left[W_1'(\Phi^{(1)}) + W_2'(\Phi^{(1)})\right] &= 0 \\ W_2'(\Phi^{(2)}) \left[W_1'(\Phi^{(2)}) + W_2'(\Phi^{(2)})\right] &= 0 . \end{aligned} \tag{6.12}$$

Assuming that $\Phi^{(1)}$ and $\Phi^{(2)}$ are diagonalizable with eigenvalues $\lambda_i^{(1)}$ and $\lambda_j^{(2)}$, one finds:

$$\begin{aligned} W_1'(\lambda_i^{(1)}) \left[W_1'(\lambda_i^{(1)}) + W_2'(\lambda_i^{(1)})\right] &= 0 \\ W_2'(\lambda_j^{(2)}) \left[W_1'(\lambda_j^{(2)}) + W_2'(\lambda_j^{(2)})\right] &= 0 . \end{aligned} \tag{6.13}$$

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On the other hand, the last row of equations in (6.11) gives:

$$\begin{aligned} \left(\lambda_j^{(2)} - \lambda_i^{(1)}\right) Q_{ji}^{(21)} &= 0 \\ \left(\lambda_i^{(1)} - \lambda_j^{(2)}\right) Q_{ij}^{(12)} &= 0\,. \end{aligned} \tag{6.14}$$

In the generic case when W_1' and W_2' have no common roots, equations (6.13) and (6.14) show that a typical classical vacuum is specified (up to permutations of indices) by choosing:

1. roots $\lambda_1^{(1)}, \ldots, \lambda_{N_1-k}^{(1)}$ of W_1',
2. roots $\lambda_1^{(2)}, \ldots, \lambda_{N_2-k}^{(2)}$ of W_2',
3. roots $\mu_1, \ldots, \mu_k$ of $W_1' + W_2'$ such that $\lambda_{N-k+m}^{(1)} = \lambda_{N-k+m}^{(2)} = \mu_m$ for $m = 1, \ldots, k$.
4. some nonzero values for $Q_{N_2-k+m,N_1-k+m}^{(21)}$ and $Q_{N_1-k+m,N_2-k+m}^{(12)}$ for $m = 1, \ldots, k$. In fact, one can set $Q_{N_1-k+m,N_2-k+m}^{(12)} = 1$ for all $m = 1, \ldots, k$ by using the gauge transformations.

Less generic vacua arise, for example, by allowing some of the μ_k to coincide. Notice that vacua with $k \neq 0$ (i.e. vacua for which some $\lambda_i^{(1)}$ coincide with some $\lambda_j^{(2)}$) are removed when imposing the condition $\gamma^{(1)} \cap \gamma^{(2)} = \emptyset$ — such vacua will not be visible unless one removes this condition on the contours. We now proceed to remove this regulator, by taking the limit in which the two curves coincide.

6.2.2 The limit of coinciding contours

To study the limit when $\gamma^{(1)}$ and $\gamma^{(2)}$ coincide, we let $\gamma^{(2)} = \gamma$, $\gamma^{(1)} = \gamma + \eta n$ (where n is the normal vector field to γ, chosen as in figure 3 of section 4) and take the positive quantity η to zero (note that this requires γ to asymptote to lines lying in the intersection of some convergence sectors of W_1 and W_2). For small η, we can use the length coordinate s of $\gamma = \gamma^{(2)}$ also as a parameter along $\gamma^{(1)}$. We then have $\lambda^{(1)}(s) = \lambda(s) + \eta n(s)$, where $\lambda(s)$ is the parameterization of γ. For $\eta \to 0^+$, the Sokhotsky formulae (4.5) give:

$$\frac{1}{\lambda_i^{(1)} - \lambda_j^{(2)}} = \mathcal{P}\frac{1}{\lambda(s_i^{(1)}) - \lambda(s_j^{(2)})} - \frac{i\pi}{\dot{\lambda}(s_j^{(2)})}\delta\left(s_i^{(1)} - s_j^{(2)}\right). \tag{6.15}$$

Therefore, the denominator in the integrand of (6.9) takes the following form:

$$\begin{aligned} \lim_{\eta\to 0^+} \frac{1}{\prod_{i,j}(\lambda_i^{(1)} - \lambda_j^{(2)})} = & \sum_{k=0}^{\min(N_1,N_2)} \sum_{\substack{1\le i_1<\cdots<i_k\le N_1 \\ 1\le j_1<\cdots<j_k\le N_2}} \sum_{\sigma\in\Sigma_k} (-i\pi)^k \frac{\delta(s_{i_1}^{(1)} - s_{j_{\sigma(1)}}^{(2)})\cdots\delta(s_{i_k}^{(1)} - s_{j_{\sigma(k)}}^{(2)})}{\dot{\lambda}(s_{j_1}^{(2)})\cdots\dot{\lambda}(s_{j_k}^{(2)})} \times \\ & \times \prod_{(i,j)\neq(i_1,j_{\sigma(1)})\cdots(i_k,j_{\sigma(k)})} \mathcal{P}\frac{1}{\lambda_i^{(1)} - \lambda_j^{(2)}} + \\ & + (\text{higher incidence terms})\,, \end{aligned} \tag{6.16}$$

where Σ_k is the group of permutations on k elements. The 'higher incidence terms' are terms involving delta-function products of the type $\delta(s_i^{(1)} - s_j^{(2)})\delta(s_i^{(1)} - s_k^{(2)}) = \delta(s_i^{(1)} -$

$s_j^{(2)})\delta(s_j^{(2)} - s_k^{(2)})$ with distinct i, j, k. Such terms can be neglected since — as we shall see in a moment — they do not contribute to the final result.

Substituting (6.16) in (6.9) gives:

$$\mathcal{Z}_{\rm lim}(\gamma) = \sum_{k=0}^{\min(N_1,N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} \mathcal{Z}_k(\gamma)\,, \tag{6.17}$$

where:

$$\mathcal{Z}_k(\gamma) = \int_\gamma d\mu_1 \cdots \int_\gamma d\mu_k \int_\gamma d\lambda_1^{(1)} \cdots \int_\gamma d\lambda_{N_1-k}^{(1)} \int_\gamma d\lambda_1^{(2)} \cdots \int_\gamma d\lambda_{N_2-k}^{(2)}\ \Delta_k\ e^{-NS_k}\,, \tag{6.18}$$

with:

$$\Delta_k := \tilde{\Delta}_k \mathcal{P} \frac{1}{\prod_{\substack{i=1\ldots N_1-k\\ j=1\ldots N_2-k}} (\lambda_i^{(1)} - \lambda_j^{(2)})}\,, \tag{6.19}$$

$$\begin{aligned}\tilde{\Delta}_k = & \prod_{i,j=1\ldots N_1-k}{}' (\lambda_i^{(1)} - \lambda_j^{(1)}) \prod_{i,j=1\ldots N_2-k}{}' (\lambda_i^{(2)} - \lambda_j^{(2)}) \prod_{i,j=1\ldots k}{}' (\mu_i - \mu_j) \times \\ & \times \prod_{\substack{i=1\ldots N_1-k\\ j=1\ldots k}} (\mu_j - \lambda_i^{(1)}) \prod_{\substack{i=1\ldots N_2-k\\ j=1\ldots k}} (\lambda_i^{(2)} - \mu_j)\end{aligned} \tag{6.20}$$

and:

$$S_k := \sum_{j=1}^{N_1-k} W_1(\lambda_j^{(1)}) + \sum_{j=1}^{N_2-k} W_1(\lambda_j^{(2)}) + \sum_{j=1}^{k} [W_1(\mu_j) + W_2(\mu_j)]\,. \tag{6.21}$$

Notice that the 'higher incidence terms' of equation (6.16) bring zero contribution to (6.17). This is due to the extra-factors of $\mu_j - \mu_k$ contributed by the two products in the numerator of the weight factor of (6.9).

Observation. Consider a quantity $H(z)$ (which depends on $\lambda_j^{(\alpha)}$ but is symmetric under separate permutations of $\lambda_1^{(1)} \ldots \lambda_{N_1}^{(1)}$ and of $\lambda_1^{(2)} \ldots \lambda_{N_2}^{(2)}$). Then its average $\langle H(z)\rangle$ has the following behavior in the limit $\eta \to 0^+$:

$$\lim_{\eta\to 0} \langle H(z)\rangle = \frac{\sum_{k=0}^{\min(N_1,N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} \mathcal{Z}_k \langle H(z)\rangle_k}{\sum_{k=0}^{\min(N_1,N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} \mathcal{Z}_k(\gamma)} := \langle H(z)\rangle_{\rm lim}\,, \tag{6.22}$$

where:

$$\langle H(z)\rangle_k = \frac{1}{\mathcal{Z}_k} \int_\gamma d\mu_1 \cdots \int_\gamma d\mu_k \int_\gamma d\lambda_1^{(1)} \cdots \int_\gamma d\lambda_{N_1-k}^{(1)} \int_\gamma d\lambda_1^{(2)} \cdots \int_\gamma d\lambda_{N_2-k}^{(2)} \Delta_k H(z) e^{-NS_k}\,. \tag{6.23}$$

Thus $\langle H(z)\rangle_{\rm lim}$ is simply a weighted average taken over the limiting ensemble.

6.2.3 Equations of motion for the limiting ensemble

Let us fix a component k of the limiting ensemble, and work with the model defined by the partition function $\mathcal{Z}_k$. Writing Δ_k as an exponential gives:

$$\mathcal{Z}_k(\gamma) = \int d\sigma_1 \cdots \int d\sigma_k \int ds_1^{(1)} \cdots \int ds_{N_1-k}^{(1)} \int ds_1^{(2)} \cdots \int ds_{N_2-k}^{(2)}\ e^{-NS_k^{\rm eff}}\,, \tag{6.24}$$

where:

$$
\begin{aligned}
S_k^{\rm eff} = S_k &- \frac{1}{N}\sum_{i,j=1\dots N_1-k}{}' \ln(\lambda_i^{(1)}-\lambda_j^{(1)}) - \frac{1}{N}\sum_{i,j=1\dots N_2-k}{}' \ln(\lambda_i^{(2)}-\lambda_j^{(2)}) - \\
&-\frac{1}{N}\sum_{i,j=1\dots k}{}' \ln(\mu_i-\mu_j) - \frac{1}{N}\sum_{\substack{i=1\dots N_1-k\\ j=1\dots k}} \ln(\mu_j-\lambda_i^{(1)}) - \frac{1}{N}\sum_{\substack{i=1\dots N_2-k\\ j=1\dots k}} \ln(\lambda_i^{(2)}-\mu_j) + \\
&+\frac{1}{N}\sum_{\substack{i=1\dots N_1-k\\ j=1\dots N_2-k}} \ln(\lambda_i^{(1)}-\lambda_j^{(2)}) - \frac{1}{N}\sum_{j=1}^{N_1-k}\ln\dot\lambda(s_j^{(1)}) - \frac{1}{N}\sum_{j=1}^{N_2-k}\ln\dot\lambda(s_j^{(2)}) - \\
&-\frac{1}{N}\sum_{j=1}^{k}\ln\dot\lambda(\sigma_j)
\end{aligned}
\tag{6.25}
$$

and we wrote $\lambda_i^{(1)} = \lambda(s_i^{(1)}), \lambda_i^{(2)} = \lambda(s_i^{(2)})$ and $\mu_i = \lambda(\sigma_i)$. Extremizing this with respect to $s_i^{(\alpha)}$ and σ_i gives the equations of motion:

$$
\begin{aligned}
&\frac{2}{N}\sum_{j=1}^{N_1-k}\frac{1}{\lambda(s_i^{(1)})-\lambda(s_j^{(1)})} - \frac{1}{N}\sum_{j=1}^{N_2-k}\frac{1}{\lambda(s_i^{(1)})-\lambda(s_j^{(2)})} + \\
&\qquad +\frac{1}{N}\sum_{j=1}^{k}\frac{1}{\lambda(s_i^{(1)})-\lambda(\sigma_j)} = W_1'(\lambda(s_i^{(1)})) - \frac{1}{N}\frac{\ddot\lambda(s_i^{(1)})}{\dot\lambda(s_i^{(1)})^2} \\
&\frac{2}{N}\sum_{j=1}^{N_2-k}\frac{1}{\lambda(s_i^{(2)})-\lambda(s_j^{(2)})} - \frac{1}{N}\sum_{j=1}^{N_1-k}\frac{1}{\lambda(s_i^{(2)})-\lambda(s_j^{(1)})} + \\
&\qquad +\frac{1}{N}\sum_{j=1}^{k}\frac{1}{\lambda(s_i^{(2)})-\lambda(\sigma_j)} = W_2'(\lambda(s_i^{(2)})) - \frac{1}{N}\frac{\ddot\lambda(s_i^{(2)})}{\dot\lambda(s_i^{(2)})^2} \\
&\frac{2}{N}\sum_{i,j=1}^{k}\frac{1}{\lambda(\sigma_i)-\lambda(\sigma_j)} + \frac{1}{N}\sum_{j=1}^{N_1-k,}\frac{1}{\lambda(\sigma_i)-\lambda(s_j^{(1)})} + \\
&\qquad +\frac{1}{N}\sum_{j=1}^{N_2-k}\frac{1}{\lambda(\sigma_i)-\lambda(s_j^{(2)})} = W_1'(\lambda(\sigma_i)) + W_2'(\lambda(\sigma_i)) - \frac{1}{N}\frac{\ddot\lambda(\sigma_i)}{\dot\lambda(\sigma_i)^2}
\end{aligned}
\tag{6.26}
$$

Let us introduce the spectral densities:

$$\rho^{(\alpha)}(s) := \frac{1}{N}\sum_{j=1}^{N_\alpha-k}\delta\left(s-s_j^{(\alpha)}\right) + \frac{1}{N}\sum_{j=1}^{k}\delta(s-\sigma_j)\,, \tag{6.27}$$

with the normalization:

$$\int ds\rho^{(\alpha)}(s) = \frac{N_\alpha}{N}\,. \tag{6.28}$$

We also introduce the traced resolvents:

$$\omega^{(\alpha)}(z) := \frac{1}{N}\sum_{j=1}^{N_\alpha-k}\frac{1}{z-\lambda(s_j^{(\alpha)})} + \frac{1}{N}\sum_{j=1}^{k}\frac{1}{z-\lambda(\sigma_j)} = \int ds\frac{\rho^{(\alpha)}(s)}{z-\lambda(s)}\,. \tag{6.29}$$

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6.2.4 The large-N Riemann surface

For every quantity ϕ, we define the large-N expansion coefficients ϕ_j as in equations (4.17). In particular, we have the planar limits $\rho_0^{(\alpha)}$ and:

$$\omega_0^{(\alpha)}(z) = \int ds \frac{\rho_0^{(\alpha)}(s)}{z-\lambda(s)} \, . \tag{6.30}$$

In the large-N limit, the eigenvalues $\lambda(s_i^{(1)})$, $\lambda(s_j^{(2)})$ and $\lambda(\sigma_k)$ accumulate on curve segments sitting along γ. The planar limits (6.30) will have cuts along three types of curvilinear intervals:

1. loci C_{13}^a resulting from a planar distribution of the eigenvalues $\lambda(s_j^{(1)})$ with $j = 1, \ldots, N_1 - k$; these will be cuts of $\omega_0^{(1)}$
2. loci C_{23}^b supporting a distribution of the eigenvalues $\lambda(s_j^{(2)})$ with $j = 1, \ldots, N_2 - k$; these give cuts of $\omega_0^{(2)}$
3. loci C_{12}^c resulting from a distribution of $\lambda(\sigma_j)$ with $j = 1, \ldots, k$; they give common cuts of $\omega_0^{(1)}$ and $\omega_0^{(2)}$.

Here a, b, c are indices counting the various occurrences of each type of cut. Note that the third type of locus can only arise from a component of the limiting ensemble for which k/N has a finite limit as $N \to \infty$. Thus cuts of type C_{12} can only develop in the large-N limit of the 'renormalized' model with $\gamma^{(1)} = \gamma^{(2)}$.

The planar limit of the equations of motion (6.26) gives:

$$\begin{aligned}
2\int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s)-\lambda(s')} - \int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s)-\lambda(s')} &= W_1'(\lambda(s)) \, , \qquad \lambda(s) \in C_{13}^a \\
2\int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s)-\lambda(s')} - \int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s)-\lambda(s')} &= W_2'(\lambda(s)) \, , \qquad \lambda(s) \in C_{23}^b \\
\int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s)-\lambda(s')} + \int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s)-\lambda(s')} &= W_1'(\lambda(s)) + W_2'(\lambda(s)) \, , \quad \lambda(s) \in C_{12}^c \, .
\end{aligned} \tag{6.31}$$

These relations act as large-N saddle point equations for the limiting ensemble (6.17). They also represent the 'quantum' version of the three branches $W_1'(\lambda) = 0$, $W_2'(\lambda) = 0$ and $W_1'(\lambda) + W_2'(\lambda) = 0$ of the classical equations of motion (6.13). It is also clear that we have:

$$\rho_0^{(1)}(s) = \rho_0^{(2)}(s) \, , \qquad \lambda(s) \in C_{12}^c \, . \tag{6.32}$$

This is the analogue of the classical relations $\lambda_i^{(1)} = \lambda_i^{(2)}$ on the corresponding branch of the moduli space (see point (3) of subsection 6.1).

As explained in appendix C, one can use (6.31) and certain partial fraction decompositions to derive the algebraic constraints:

$$\begin{aligned}
\omega_0^{(1)}(z)^2 - \omega_0^{(1)}(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2 - W_1'(z)\omega_0^{(1)}(z) - W_2'(z)\omega_0^{(2)}(z) + f_0^{(1)}(z) + f_0^{(2)}(z) &= 0 \\
\omega_0^{(1)}(z)^2\omega_0^{(2)}(z) - W_1'(z)\left[\omega_0^{(1)}(z)^2 + f_0^{(1)}(z) - W_1'(z)\omega_0^{(1)}(z)\right] + g_0^{(1)}(z) - (1 \leftrightarrow 2) &= 0
\end{aligned} \tag{6.33}$$

where $f_0^{(\alpha)}, g_0^{(\alpha)}$ are polynomials defined through:

$$f_0^{(\alpha)}(z) := \int ds \rho_0^{(\alpha)}(s) \frac{W_\alpha'(z) - W_\alpha'(\lambda(s))}{z - \lambda(s)} \tag{6.34}$$

and

$$\begin{aligned} g_0^{(1)}(z) &:= \int d\alpha \int d\beta \left[W_1'(z) - W_1'(\lambda(\alpha))\right] \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} \\ g_0^{(2)}(z) &:= \int d\alpha \int d\beta \left[W_2'(z) - W_2'(\lambda(\alpha))\right] \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} . \end{aligned} \tag{6.35}$$

Note that the normalization conditions (6.28) imply:

$$\text{lcoeff}\left(f_0^{(\alpha)}\right) = \frac{N_\alpha}{N} \deg(W_\alpha) \text{lcoeff}(W_\alpha) . \tag{6.36}$$

where lcoeff$(\ldots)$ denotes the leading coefficient. Writing:

$$\begin{aligned} \omega_0^{(1)}(z) &= u_1(z) + t_1(z) \\ \omega_0^{(2)}(z) &= -u_2(z) + t_2(z) , \end{aligned} \tag{6.37}$$

where:

$$\begin{aligned} t_1(z) &:= \frac{2W_1'(z) + W_2'(z)}{3} \\ t_2(z) &:= \frac{2W_2'(z) + W_1'(z)}{3} \end{aligned} \tag{6.38}$$

brings the constraints (6.33) to the form:

$$\begin{aligned} u_1(z)^2 + u_1(z)u_2(z) + u_2(z)^2 &= p(z) \\ u_1(z)^2 u_2(z) + u_1(z)u_2(z)^2 &= -q(z) , \end{aligned} \tag{6.39}$$

with:

$$\begin{aligned} p(z) &:= t_1(z)^2 - t_1(z)t_2(z) + t_2(z)^2 - f_0^{(1)}(z) - f_0^{(2)}(z) \\ q(z) &:= -t_1(z)t_2(z)\left[t_1(z) - t_2(z)\right] - t_1(z)f_0^{(2)}(z) + t_2(z)f_0^{(1)}(z) - g_0(z) . \end{aligned} \tag{6.40}$$

In the last equation, we introduced the polynomial $g_0(z) = g_0^{(1)}(z) - g_0^{(2)}(z)$ which has the following explicit form in terms of matrix model data:

$$\begin{aligned} g_0(z) = &\int d\alpha \int d\beta \left[W_1'(z) - W_1'(\lambda(\alpha))\right] \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} - \\ &- \int d\alpha \int d\beta \left[W_2'(z) - W_2'(\lambda(\alpha))\right] \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} \end{aligned} \tag{6.41}$$

Defining $u_3(z) := -u_1(z) - u_2(z)$, identities (6.39) can be recognized as the Viete relations of the cubic:

$$u^3 - p(z)u - q(z) = 0 \tag{6.42}$$

when the left hand side is viewed as a polynomial in u. This shows that $u_1(z)$, $u_2(z)$ and $u_3(z)$ are the three branches of the affine algebraic curve (6.42). This is the precise form of the curve suggested in [28] (where the explicit form of the polynomials p, q in terms of matrix model data was not given). Note that the left hand side of (6.42) can also be written as:

$$(u-u_1)(u-u_2)(u-u_3) = (u+t_1)(u-t_2)(u-t_1+t_2)+(f_0^{(1)}+f_0^{(2)})u+t_1f_0^{(2)}-t_2f_0^{(1)}+g_0\,. \tag{6.43}$$

Studying the behavior of u_j near the branch cuts of (6.42) allows one to identify these as the loci C_{13}^a, C_{23}^b and C_{12}^c where the eigenvalues accumulate; then the jump equations across these cuts can be seen to be equivalent with the planar equations of motion (6.31). Below, we shall use this reasoning in order to give a reconstruction theorem for the holomorphic A_2 model, similar to the one we found in section 4 for the holomorphic one-matrix model.

6.2.5 The reconstruction theorem

Let us start with a curve of form (6.42) with p, q given by (6.40), where $f_0^{(\alpha)}$ and g_0 are complex polynomials of degree $n-2$ subject to the constraints (6.36). Given such data, we show how one can construct a curve γ and distributions $\rho_0^{(\alpha)}(s)$ along this curve such that relations (6.28), (6.30), (6.31), (6.32) and (6.34), (6.41) hold.

Using expression (6.43), one finds the following asymptotic behavior for large z:

$$\begin{aligned} u_1(z) &= -t_1 + \frac{\mathrm{lcoeff}(f_0^{(1)})}{deg(W_1)\mathrm{lcoeff}(W_1)}\frac{1}{z} + O\left(\frac{1}{z^2}\right) \\ u_2(z) &= +t_2 + \frac{\mathrm{lcoeff}(f_0^{(2)})}{deg(W_2)\mathrm{lcoeff}(W_2)}\frac{1}{z} + O\left(\frac{1}{z^2}\right). \end{aligned} \tag{6.44}$$

In particular, we can use these asymptotic forms in order to fix the indexing of branches for (4.23) (i.e. u_1 is the branch which asymptotes to $-t_1$, while u_2 is the branch which asymptotes to $+t_2$).

When viewed as a triple cover of the z-plane, the curve (6.42) has three types of cuts, namely those connecting the pairs of branches $(1,3)$, $(2,3)$ and $(1,2)$. Denote these cuts by C_{13}^a, C_{23}^b and C_{12}^c. Picking γ to contain all cuts, we let $\omega_0^{(1)}$ and $\omega_0^{(2)}$ be given by relations (6.37) and define:

$$\begin{aligned} \rho_0^{(1)}(s) &:= \frac{1}{2\pi i}\dot\lambda(s)\left[\omega_0^{(1)}(\lambda(s)-i0)-\omega_0^{(1)}(\lambda(s)+i0)\right], \quad \text{for } \lambda(s)\in C_{13}^a \\ \rho_0^{(2)}(s) &:= \frac{1}{2\pi i}\dot\lambda(s)\left[\omega_0^{(2)}(\lambda(s)-i0)-\omega_0^{(2)}(\lambda(s)+i0)\right], \quad \text{for } \lambda(s)\in C_{23}^b\,, \\ \rho^{(1)}(s)=\rho_0^{(2)}(s) &:= \frac{1}{2\pi i}\dot\lambda(s)\left[\omega_0^{(1)}(\lambda(s)-i0)-\omega_0^{(1)}(\lambda(s)+i0)\right] \\ &= \frac{1}{2\pi i}\dot\lambda(s)\left[\omega_0^{(2)}(\lambda(s)-i0)-\omega_0^{(2)}(\lambda(s)+i0)\right], \quad \text{for } \lambda(s)\in C_{12}^c\,. \end{aligned} \tag{6.45}$$

The very last equality in these relations follows by using (6.37) and the fact that $u_3 = -u_1-u_2$ is continuous across cuts of type C_{12}. In particular, this means that (6.32) holds. Extending $\rho_0^{(\alpha)}$ by zero to the entire curve γ, relations (6.45) and the Sokhotsky formulae immediately imply that equations (6.30) hold. In turn, these relations, the asymptotic behavior (6.44) and the constraints (6.36) on the leading coefficients imply the normalization conditions (6.28).

We next notice that:

$$\begin{aligned} u_1(z) - u_3(z) &= 2u_1(z) + u_2(z) = 2\omega_0^{(1)}(z) - \omega_0^{(2)}(z) - W_1'(z) \\ u_2(z) - u_3(z) &= u_1(z) + 2u_2(z) = \omega_0^{(1)}(z) - 2\omega_0^{(2)}(z) + W_2'(z) \\ u_1(z) - u_2(z) &= \omega_0^{(1)}(z) + \omega_0^{(2)}(z) - (W_1'(z) + W_2'(z)), \end{aligned} \tag{6.46}$$

where we used relations (6.37) and (6.38). Since a cut of type $C_{\alpha\beta}$ connects the branches u_α and u_β, we have the jump equations:

$$\begin{aligned} & u_\alpha(\lambda \pm 0n(\lambda)) = u_\beta(\lambda \mp 0n(\lambda)) \Rightarrow \\ & [u_\alpha(\lambda + 0n(\lambda)) - u_\beta(\lambda + 0n(\lambda))] + [u_\alpha(\lambda - 0n(\lambda)) - u_\beta(\lambda - 0n(\lambda))] = 0 \end{aligned} \tag{6.47}$$

along such a cut. Combining this with relations (6.46) and using equations (6.30) and the Sokhotsky formulae (4.5) shows that $\rho_0^{(\alpha)}$ satisfy the planar equations of motion (6.31).

To prove that (6.34) and (6.41) hold, one can now simply repeat the original argument (found in appendix C) leading to the large-N curve (6.42); this is possible since we already showed that all assumptions of that argument (namely relations (6.30), (6.31), (6.32)) hold. This shows that equations (6.39) and (6.40) must hold with polynomials f, g defined by relations (6.34) and (6.41). Since by hypothesis the same relations hold with the original polynomials $f_0^{(\alpha)}$ and g_0, it immediately follows that the latter can indeed be expressed in the form (6.34) and (6.41). This concludes the proof that the 'renormalized' holomorphic A_2 model indeed probes the whole moduli space of (6.42) with the constraints (6.36).

Observation The third 'branch' of the planar equations of motion (6.31) is only allowed in the 'renormalized' model described by the limiting ensemble (6.17). If one works with the regularized model instead (the original model for which $\gamma^{(1)}$ and $\gamma^{(2)}$ are disjoint), then cuts of type C_{12}^c are not allowed. Indeed, such cuts connect the branches u_1 and u_2, and thus are cuts for both $\omega_0^{(1)}$ and $\omega_0^{(2)}$, which would require $C_{12}^c \subset \gamma^{(1)} \cap \gamma^{(2)}$, in contradiction with the regularization condition $\gamma^{(1)} \cap \gamma^{(2)} = \emptyset$. Hence the regularized model can only probe that part of the moduli space of (6.42) for which all cuts of type C_{12}^c are reduced to double points. This is similar to what happens for the hermitean A_2 model, as discussed at the beginning of this section. In particular, the regularized representation (6.9) cannot capture the entire family of curves (6.42), and, in fact, can only describe a subvariety in the space of all allowed polynomials $f_0^{(\alpha)}$ and $g_0^{(\alpha)}$. To explore the full moduli space, one *must* consider the limiting ensemble (6.17). In the context of the Dijkgraaf-Vafa conjecture, the dual field theory is an $N = 1$ supersymmetric field theory derived from an A_2 quiver (such theories have been studied in [22, 23]). This field theory is certainly allowed to explore the branch whose classical description is given by $W_1'(\lambda) + W_2'(\lambda) = 0$. Therefore, consideration of the limiting ensemble (6.17) of the holomorphic model is required by the conjecture of [1] as applied to the A_2 quiver field theories of [22]. A similar analysis can be performed for general ADE models, with the same conclusion.

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A. Integration over gauge group orbits for the holomorphic one-matrix model

In this appendix, we show how the eigenvalue representation (2.14) results from the matrix integral (2.7).

A.1 Orbit decomposition of $\mathcal{M}$

Let Δ be the set of diagonal $N \times N$ complex matrices D with distinct eigenvalues:

$$\Delta := \{D = \mathrm{diag}(\lambda_1 \dots \lambda_N) | \lambda_j \in \mathbb{C} \text{ and } \prod_{\mathrm{i} \neq \mathrm{j}} (\lambda_\mathrm{i} - \lambda_\mathrm{j}) \neq 0\} \tag{A.1}$$

and fix a fundamental domain Δ_0 for the obvious action of the permutation group Σ_N on Δ. Then relation (2.2) gives the orbit decomposition:

$$\mathcal{M} = \sqcup_{D \in \Delta_0} \mathcal{O}_D \,, \tag{A.2}$$

where $\mathcal{O}_D$ is the $\mathrm{GL}(N, \mathbb{C})$-orbit of a matrix $D \in \Delta_0$ under the similarity action (2.8).

The stabilizer of each $D \in \Delta_0$ in $\mathrm{GL}(N, \mathbb{C})$ is the subgroup $T_N \approx (\mathbb{C}^*)^N$ consisting of diagonal matrices. Hence each orbit has the form:

$$\mathcal{O}_D = H := \mathrm{GL}(N, \mathbb{C}) / T_N \,, \tag{A.3}$$

where the homogeneous space H has dimension $N^2 - N$ (here T_N acts on $\mathrm{GL}(N, \mathbb{C})$ from the right, i.e. $S \to ST$ for $S \in \mathrm{GL}(N, \mathbb{C})$ and $T \in T_N$). The orbit decomposition (A.2) takes the form:

$$\mathcal{M} = H \times \Delta_0 \,. \tag{A.4}$$

A.2 Decomposition of w

Let:

$$w_\Delta = \prod_{j=1}^{N} d\lambda_j \tag{A.5}$$

be the translation-invariant holomorphic volume form on Δ and:

$$w_H = \wedge_{i \neq j} \omega_{ij} \tag{A.6}$$

be the left-invariant holomorphic volume form on the complex homogeneous space H, where $\omega = S^{-1} dS$ is the matrix whose elements give a basis of left-invariant holomorphic one-forms on $\mathrm{GL}(N, \mathbb{C})$.

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If $l_S(S') := SS'$ is the left action of $\mathrm{GL}(N,\mathbb{C})$ on H, then w_H satisfies:

$$(l_S)^* w_H = w_H \, , \qquad S \in \mathrm{GL}(N,\mathbb{C}) \, . \tag{A.7}$$

Using the projections π_H and π_Δ of $\mathcal{M} = H \times \Delta_0$ onto its two factors, we define a holomorphic N^2-form on $\mathcal{M}$ by:

$$w_0 := \pi_H^*(w_H) \wedge \pi_\Delta^*(w_\Delta) \, . \tag{A.8}$$

For dimension reasons, this must be related to w through:

$$w = f w_0 \, , \tag{A.9}$$

where $f(M)$ is a holomorphic function on $\mathcal{M}$. Using the left-invariance (A.7) of w_H, it is easy to check that w_0 is invariant[7] under the action (2.8):

$$\tau(S)^*(w_0) = w_0 \, . \tag{A.10}$$

Using $\mathrm{GL}(N,\mathbb{C})$ invariance of w and w_0, relation (A.9) implies that f is invariant:

$$f(SMS^{-1}) = f(M) \, . \tag{A.11}$$

In particular, we have $f(M) = f(D)$ if $SMS^{-1} = D$ with D diagonal. It thus suffices to determine the values of f for diagonal matrices D.

For this, we first describe the the cotangent space to $\mathcal{M}$ at the points of Δ_0 (where Δ_0 is viewed as a subset of $\mathcal{M}$). In the vicinity of Δ_0, we can write $M = SDS^{-1} \approx (1+A)D(1-A) \approx D + [A,D]$, where $S = e^A \approx 1 + A$ with A an infinitesimal generator of $\mathrm{GL}(N,\mathbb{C})$. Therefore, at a point D of Δ_0, we have:

$$dM = dD + [dA, D] \Longleftrightarrow dM_{ij} = \delta_{ij} d\lambda_i + (\lambda_j - \lambda_i) dA_{ij} \, . \tag{A.12}$$

Note that there is no $[A, dD]$ piece in the right hand side of this relation, since we compute the form dM at the point D (where $A = 0$).

Noticing that $w_H = \wedge_{i \neq j} dA_{ij}$ at such a point, relation (A.12) gives the form of w at $M = D$:

$$w = (-1)^{N^2(N-1)/2} \wedge_{i,j} dM_{ij} = (-1)^{N^2(N-1)/2} \left[\prod_{i \neq j} (\lambda_i - \lambda_j) \right] w_0 \, . \tag{A.13}$$

Comparing with (A.9), we find that f is given by the usual Vandermonde determinant:

$$f(M) = f(D) = \prod_{i \neq j} (\lambda_i - \lambda_j) \, . \tag{A.14}$$

Combining this with (A.9) gives the following expression for the holomorphic measure:

$$w = (-1)^{N^2(N-1)/2} \prod_{i \neq j} (\lambda_i - \lambda_j) w_0 \, , \tag{A.15}$$

where λ_i are the eigenvalues of the matrix M at which we evaluate w.

[7] To check this, notice that $\pi_H \circ \tau(S) = l_S \circ \pi_H$.

A.3 The eigenvalue representation

The $\mathrm{GL}(N,\mathbb{C})$ invariance of the action, relation (A.15) and the decomposition (A.8) allow us to perform the integral over the gauge-group variables in the partition function (2.7):

$$\tilde{\mathcal{Z}}_N(\gamma,t) = \frac{1}{\mathcal{N}}(-1)^{N^2(N-1)/2}\,\mathrm{hvol}(H)\int_{\Delta_0(\gamma)}\prod_{j=1}^{N} d\lambda_j \prod_{i\neq j}(\lambda_i-\lambda_j)e^{-N\sum_{j=1}^{N}W(\lambda_j)} \quad \text{(A.16)}$$

where $\mathrm{hvol}(H) = \int_H \omega_H$ is the 'holomorphic volume' of H and:

$$\Delta_0(\gamma) = \{D = \mathrm{diag}(\lambda_1\dots\lambda_N) \in \Delta_0 | \lambda_j \in \gamma\,,\ \forall j = 1,\dots,N\}\,. \quad \text{(A.17)}$$

Since this relation holds for *any* choice of fundamental domain Δ_0, we can write (A.16) in the form:

$$\tilde{\mathcal{Z}}_N(\gamma,t) = \frac{1}{\mathcal{N}}(-1)^{N^2(N-1)/2}\frac{1}{N!}\,\mathrm{hvol}(H)\mathcal{Z}_N(\gamma,t)\,, \quad \text{(A.18)}$$

where:

$$\mathcal{Z}_N(\gamma,t) = \int_{\Delta(\gamma)}\prod_{j=1}^{N} d\lambda_j \prod_{i\neq j}(\lambda_i-\lambda_j)e^{-N\sum_{j=1}^{N}W(\lambda_j)} \quad \text{(A.19)}$$

with:

$$\Delta(\gamma) = \left\{D = \mathrm{diag}(\lambda_1\dots\lambda_N)|\lambda_j\in\gamma\,,\ \forall j=1,\dots,N \text{ and } \prod_{i\neq j}(\lambda_i-\lambda_j)\neq 0\right\} \quad \text{(A.20)}$$

This gives the eigenvalue representation (2.14).

Observation. When writing the representation (2.14), we have tacitly extended the integral to allow for colliding eigenvalues $\lambda_i = \lambda_j$ (this is certainly allowed, since the integrand of (A.19) is well-behaved there). This amounts to treating non-diagonalizable matrices by a point-splitting prescription, and can be formulated in more detail by working with:

$$\Delta_\epsilon := \{D = \mathrm{diag}(\lambda_1\dots\lambda_N)|\lambda_j\in\mathbb{C} \text{ and } |\lambda_\mathrm{i}-\lambda_\mathrm{j}| > \epsilon \qquad \text{for i} \neq \text{j}\} \quad \text{(A.21)}$$

instead of Δ and with a similar modification $\mathcal{M}_\epsilon$ of $\mathcal{M}$. Then one defines the partition function as the limit $\epsilon \to 0^+$ of the regularized partition function obtained in this manner. It is easy to adapt the derivation given above in order to include explicitly such a regulator. The result, however, is the same as (2.14), because the integrand of (A.19) is well-behaved when eigenvalues come close to each other.

B. Example of the relevance of convergence sectors: the case of a cubic potential

Consider the potential:

$$W(z) = t_3z^3 + t_2z^2\,. \quad \text{(B.1)}$$

This example appeared in the paper [8], where it was used to carry out a one-loop test of the Dijkgraaf-Vafa conjecture. As in [8], we assume for simplicity that t_2 and t_3 are real and positive, and write them as $t_2 = m/2$ and $t_3 = g/3$ with positive m and g. The potential has two local extrema along the real axis, namely a local minimum at $a_1 = 0$ and a local maximum at $a_2 = -2t_2/3t_3 = -m/g < 0$. Also note that $W(z)$ approaches $\pm\infty$ as z approaches $\pm\infty$ along the real axis.

B.1 Summary of the procedure of [8]

The paper [8] follows [1] by formulating a conjecture mapping a one-matrix model based on the potential (B.1) to the closed topological B-model on a certain non-compact Calabi-Yau space. When testing this, [8] performs a perturbative expansion of this model around the two extrema a_1 and a_2, thereby re-writing the theory as a a two-matrix model for matrices M_1, M_2 with cubic potentials $W_1(M_1)$ and $W_2(M_2)$ and a logarithmic interaction term $W_{\rm int}(M_1, M_2)$. Since the point a_2 is a local *maximum* for the original potential W, this would produce a negative definite quadratic term in $W_2(M_2)$, if one starts with a hermitean matrix model, thereby leading to an ill-defined perturbation expansion. To cure this problem, [8] proposes that one should take M_1 to be anti-hermitean and M_2 to be hermitean. With this redefinition, the authors of [8] compute the first few perturbative terms and find agreement with the one-loop computation of the Kodaira-Spencer theory of the Calabi-Yau dual. We now show how this pragmatic procedure can be recovered in the holomorphic framework.

B.2 Justification in terms of holomorphic matrix models

Remember that a hermitean matrix model based on the cubic potential (B.1) is ill-defined. This is due to the exponential increase of the integrand for $\lambda_j \to -\infty$. On the other hand, the holomorphic matrix model leads to a meaningful integral, provided that one chooses appropriate asymptotic sectors.

Note that our definition (2.7) uses the weight $\prod_{i\neq j}(\lambda_i - \lambda_j)$ rather than the weight $\prod_{i<j}(\lambda_i - \lambda_j)^2$ which is used in [8]. In this appendix, we shall temporarily switch to the representation:

$$\mathcal{Z}_N(\gamma, t) = \int_\gamma d\lambda_1 \cdots \int_\gamma d\lambda_N \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N\sum_{j=1}^N W(\lambda_j)} \tag{B.2}$$

which differs from our convention (2.14) by a sign prefactor of $(-1)^{N(N-1)/2}$. This avoids some annoying sign factors when comparing with [8].

Thus we start with the partition function (B.2) for the potential (B.1). Since $t_3 > 0$, this model has $\theta_3 = 0$ and the asymptotic sectors shown in figure 7. We claim that the correct partition function relevant for the work of [8] is $\mathcal{Z}(1, 0, t)$, associated with the asymptotic sectors $k_- = 1$ and $k_+ = 0$. This is convergent by the analysis of section 3.

In this set-up, the prescription of [8] can be recovered as follows (we shall neglect the gauge group volume prefactors, which are irrelevant here). Since the partition function (B.2) depends only on the asymptotic sectors of γ (namely $\nu_- \in A_1$ and $\nu_+ \in A_0$), we are free to choose this curve as shown in figure 7. This asymptotes to some line d_- with tangent $\nu_- \in A_1$ for $t \to -\infty$, then follows a vertical line through the point $a_2 = -m/g$,

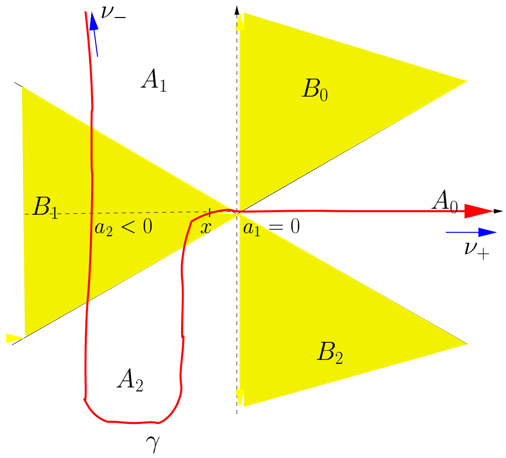

Figure 7: Convergence sectors for the case *deg* $W = 3$, $\theta_3 = 0$ and a good choice of contour belonging to the sector $(k_-, k_+) = (1, 0)$.

and finally loops back to touch (and then follow) the real axis at some point x lying in between $a_2 = -m/g$ and $a_1 = 0$. Thus an eigenvalue λ sitting on γ will be imaginary if it lies close to a_2 and real if it lies close to a_0, thereby *naturally* implementing the requirement of [8]. The curve segments along γ around the points a_1 and a_2 for which λ has these properties can be maximized by making the 'well' at the bottom of this curve to be very thin (i.e. take x to be very close to a_2) and very deep.

Following [8], we now expand the integral (B.2) (with γ chosen as above) around the saddle point configurations:

$$\lambda_i^{(0)} = a_1 = 0 \quad \text{for } i = 1, \ldots, N_1\,, \qquad \lambda_j^{(0)} = a_2 = -\frac{m}{g} \quad \text{for } j = N_1 + 1, \ldots, N \tag{B.3}$$

with N_1 some integer in the set $\{1, \ldots, N\}$. Let $N_2 := N - N_1$. This gives:

$$\mathcal{Z}(t, \gamma) = \sum_{N_1 + N_2 = N} \frac{N!}{N_1! N_2!} \int_\gamma d\mu_1 \cdots \int_\gamma d\mu_{N_1} \int_\gamma d\nu_1 \cdots \int_\gamma d\nu_{N_2} \Delta(\mu, \nu) e^{-NS} \tag{B.4}$$

where we wrote $\lambda_i = \lambda_i^{(0)} + \mu_i$, $\lambda_{N_1+j} = \lambda_{N_1+j}^{(0)} + \nu_j$ and:

$$\begin{aligned} \Delta(\mu, \nu) &= \prod_{1 \leq i_1 < i_2 \leq N_1} (\mu_{i_1} - \mu_{i_2})^2 \prod_{1 \leq j_1 < j_2 \leq N_1} (\nu_{j_1} - \nu_{j_2})^2 \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left(\mu_i - \nu_j + \frac{m}{g} \right)^2 \\ S &= \sum_{i=1}^{N_1} W(\mu_i) - \sum_{j=1}^{N_2} W(-\nu_j) + \frac{m^3}{6g^2} N_2\,. \end{aligned} \tag{B.5}$$

Treating μ_i and ν_j as small fluctuations, we naturally have $\mu_i \in i\mathbb{R}$ and $\nu_j \in \mathbb{R}$, since the eigenvalues λ lying along γ are imaginary in the vicinity of a_2 and real in the vicinity of a_1. Writing the last term in Δ as an exponential, one obtains a logarithmic interaction potential:[8]

$$W_{\rm int} = 2N_1N_2 \ln\frac{m}{g} + 2\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \ln\left[1+\frac{g}{m}(\mu_i-\nu_j)\right] \tag{B.6}$$

and one-matrix potentials $W_1(\mu) = W(\mu)$ and $W_2(\nu) = -W(-\nu)$, which allows one to write the result as an interacting two-matrix model [8]:

$$Z = \sum_{N_1+N_2=N} \frac{N!}{N_1!N_2!}\int_{M_1^+=-M_1} dM_1 \int_{M_2^+=+M_2} dM_2 e^{-N\,{\rm tr}[W_1(M_1)+W_2(M_2)]+W_{\rm int}(M_1,M_2)}, \tag{B.7}$$

where the first integral is over anti-hermitean matrices while the second integral is over hermitean matrices. Note that anti-Hermiticity of M_1 arises *naturally* in the holomorphic set-up. This gives a conceptual justification for the procedure of [8]. Note also that we have provided a non-perturbative construction of the model involved in that work: it is simply the holomorphic matrix model with potential (B.1), considered in the 'phase' $(k_-, k_+) = (1, 0)$.

C. Derivation of the planar constraints for the A_2 model

Let us show how the non-hyperelliptic Riemann surface expected from the observations of [28] arises in the holomorphic A_2 model. For this, we derive two algebraic constraints which hold in the planar limit, as a consequence of the planar equations of motion (6.31).

To derive the equations of interest, we shall use the partial fraction decompositions:

$$\frac{1}{(z-u)(z-v)} = \frac{1}{u-v}\left[\frac{1}{z-u} - \frac{1}{z-v}\right] \tag{C.1}$$

and:

$$\begin{aligned}\frac{1}{(z-u)(z-v)(z-w)} = &\frac{1}{(u-v)(u-w)}\frac{1}{z-u} + \frac{1}{(v-u)(v-w)}\frac{1}{z-v} + \\ &+\frac{1}{(w-u)(w-v)}\frac{1}{z-w}\,.\end{aligned} \tag{C.2}$$

C.1 The first constraint

Using (C.1), one finds:

$$\begin{aligned}\omega_0^{(1)}(z)^2 &= 2\int ds\int ds'\frac{\rho_0^{(1)}(s)\rho_0^{(1)}(s')}{\lambda(s)-\lambda(s')}\frac{1}{z-\lambda(s)}\\ \omega_0^{(2)}(z)^2 &= 2\int ds\int ds'\frac{\rho_0^{(2)}(s)\rho_0^{(2)}(s')}{\lambda(s)-\lambda(s')}\frac{1}{z-\lambda(s)}\\ \omega_0^{(1)}(z)\omega_0^{(2)}(z) &= \int ds\int ds'\left[\frac{\rho_0^{(1)}(s)\rho_0^{(2)}(s')}{\lambda(s)-\lambda(s')}\frac{1}{z-\lambda(s)} + \frac{\rho_0^{(2)}(s)\rho_0^{(1)}(s')}{\lambda(s)-\lambda(s')}\frac{1}{z-\lambda(s)}\right].\end{aligned} \tag{C.3}$$

[8]The authors of [8] further expand the logarithm as a power series in $\frac{g}{m}(\mu_i-\nu_j)$, a procedure which is justified if $|\frac{g}{m}(\mu_i-\nu_j)|<1$.

Combining these equations gives:

$$\omega_0^{(1)}(z)^2-\omega_0^{(1)}(z)\omega_0^{(2)}(z)+\omega_0^{(2)}(z)^2=\int ds\rho_0^{(1)}(s)\frac{W_1'(\lambda(s))}{z-\lambda(s)}+\int ds\rho_0^{(2)}(s)\frac{W_2'(\lambda(s))}{z-\lambda(s)}. \quad (C.4)$$

To arrive at this relation, we decomposed the integrals over ds in (C.3) into the pieces corresponding to the cuts C_{13}^a, C_{23}^b and C_{12}^c. Then equation (C.4) results upon combining these pieces appropriately and performing the s' integral by using the planar equations of motion (6.31) and relations (6.32). We next write (C.4) in the form:

$$\omega_0^{(1)}(z)^2-\omega_0^{(1)}(z)\omega_0^{(2)}(z)+\omega_0^{(2)}(z)^2-W_1'(z)\omega_0^{(1)}(z)-W_2'(z)\omega_0^{(2)}(z)+f_0^{(1)}(z)+f_0^{(2)}(z)=0\,, \quad (C.5)$$

where we used the planar equations of motion (6.31) and the definition of $\omega_0^{(\alpha)}(z)$ and we introduced the polynomials:

$$\begin{aligned} f_0^{(1)}(z) &:= \int ds\rho_0^{(1)}(s)\frac{W_1'(z)-W_1'(\lambda(s))}{z-\lambda(s)} \\ f_0^{(2)}(z) &:= \int ds\rho_0^{(2)}(s)\frac{W_2'(z)-W_2'(\lambda(s))}{z-\lambda(s)}\,. \end{aligned} \quad (C.6)$$

C.2 The second constraint

To derive the second constraint, we use (C.2) to compute:

$$\begin{aligned} \omega_0^{(1)}(z)^2\omega_0^{(2)}(z) &= \int d\alpha\int d\beta\int d\gamma\frac{2\rho_0^{(1)}(\alpha)\rho_0^{(1)}(\beta)\rho_0^{(2)}(\gamma)}{(\lambda(\alpha)-\lambda(\beta))(\lambda(\alpha)-\lambda(\gamma))(z-\lambda(\alpha))}+ \\ &+\int d\alpha\int d\beta\int d\gamma\frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)\rho_0^{(1)}(\gamma)}{(\lambda(\alpha)-\lambda(\beta))(\lambda(\alpha)-\lambda(\gamma))(z-\lambda(\alpha))}\,, \end{aligned} \quad (C.7)$$

where we used redefinitions of (α,β,γ) by permutations to bring the right hand side to a convenient form. Combining this with the equation obtained by permuting the indices $1\leftrightarrow 2$ gives:

$$\omega_0^{(1)}(z)^2\omega_0^{(2)}(z)-\omega_0^{(1)}(z)\omega_0^{(2)}(z)^2=\int d\alpha\int d\beta W_1'(\lambda(\alpha))\frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha)-\lambda(\beta))(z-\lambda(\alpha))}-(1\leftrightarrow 2)\,, \quad (C.8)$$

where we used the planar equations of motion (6.31) to perform the integral over β. Defining the polynomials:

$$\begin{aligned} g_0^{(1)}(z) &:= \int d\alpha\int d\beta\left[W_1'(z)-W_1'(\lambda(\alpha))\right]\frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha)-\lambda(\beta))(z-\lambda(\alpha))}\,, \\ g_0^{(2)}(z) &:= \int d\alpha\int d\beta\left[W_2'(z)-W_2'(\lambda(\alpha))\right]\frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha)-\lambda(\beta))(z-\lambda(\alpha))}\,, \end{aligned} \quad (C.9)$$

we find:

$$\omega_0^{(1)}(z)^2\omega_0^{(2)}(z)-\omega_0^{(1)}(z)\omega_0^{(2)}(z)^2+g_0^{(1)}(z)-g_0^{(2)}(z)-W_1'(z)U_1(z)+W_2'(z)U_2(z)=0\,, \quad (C.10)$$

where:

$$U_1(z) := \int d\alpha \int d\beta \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha)-\lambda(\beta))(z-\lambda(\alpha))}$$
$$U_2(z) := \int d\alpha \int d\beta \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha)-\lambda(\beta))(z-\lambda(\alpha))}\,. \tag{C.11}$$

Using the equations of motion (6.31), these quantities can be written:

$$U_1(z) = f_0^{(1)}(z) - W_1'(z)\omega_0^{(1)}(z) + \omega_0^{(1)}(z)^2$$
$$U_2(z) = f_0^{(2)}(z) - W_2'(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2\,. \tag{C.12}$$

Therefore, equation (C.10) becomes:

$$\omega_0^{(1)}(z)^2\omega_0^{(2)}(z) - W_1'(z)[\omega_0^{(1)}(z)^2 - W_1'(z)\omega_0^{(1)}(z) + f_0^{(1)}(z)] + g_0^{(1)}(z) - (1 \leftrightarrow 2) = 0\,. \tag{C.13}$$

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